國立中山大學應用數學系 碩士論文 卡式乘積圖的控制數 研究生：王文 撰 101學年度

# 國立中山大學應用數學系

## 碩士論文

Department of Applied Mathematics

National Sun Yat-sen University

Master Thesis

# 卡式乘積圖的控制數

# Domination number of Cartesian product of graphs


研究生：王文

Wen Wang

指導教授：朱緒鼎 博士

Dr. Xuding Zhu


中華民國 102 年 8 月

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# 國立中山大學研究生學位論文審定書

本校應用數學系碩士班

研究生王　文（學號：M972040025）所提論文

卡式乘積圖的控制數

Domination number of Cartesian product of graphs

於中華民國 102 年 7 月 29 日經本委員會審查並舉行口試，符合碩士學位論文標準。

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# 謝　辭

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# 摘　要


本論文探討著名的 Vizing 猜想：任意兩個圖的卡氏積圖其控制數大於或等於兩個圖的控制數的乘積。我們證明了如果圖 G 有 k 個完全子圖$G_1$、$G_2$.....$G_k$ 使得 $\square_G(\bigcup_{i=1}^{k} G_i)$ 等於 k，則對於任意圖 H，兩個圖 G 和 H 的卡氏積圖的控制數一定會大於或等於 k 個 H 的控制數。



**關鍵字 ： Vizing 猜想，卡氏積，控制數。**

Domination Number of Cartesian Product of Graphs

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Submitted to the Faculty

of

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by

Wen Wang

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of

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# Domination Number of Cartesian Product of Graphs

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**Abstract**

For a graph $G$, $\gamma(G)$ is the domination number of $G$. Vizing [2] conjectured that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ for any graph $G$ and $H$, where $G\Box H$ is the Cartesian product of graphs $G$ and $H$. Clark and Suen [1] proved that $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$ for any graphs $G$ and $H$. Barcalkin and German [5] proved that Vizing's conjecture holds for some specific family of graphs. We combine both of their approaches and prove that if $G$ has $k$ disjoint complete subgraphs $G_1, G_2, \ldots G_k$ and $\gamma_G(\bigcup_{i=1}^{k} G_i) = k$, then $\gamma(G\Box H) \geq k\gamma(H)$.

Keywords: Vizing Conjecture, Cartesian Product, domination number.

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# Chapter 1

# Introduction

One problem studied extensively in graph theory is graph invariants for product graphs. There are various graph invariants and various graph products. In 1996, Nowakowski and Rall [4] explored twelve graph invariants on ten associative graph products whose edge structure depends on that of both factors. A natural question is how an invariant of the product graphs is related to the corresponding invariants of the factor graphs. For some invariants and products, there are simple relations between the invariant of a product graph and the corresponding invariants of the factor graphs. An example of this situation is that the chromatic number of the Cartesian product of two graphs is the maximum of the chromatic numbers of the two factor graphs. In some cases, there are upper and lower bounds for the invariants of product graphs in terms of the corresponding invariants of factor graphs. For example there are upper bounds for the independence number of the direct product of graphs in terms of the independence numbers of factor graphs. In general, no exact formula is known for the product graph in terms of the independence numbers of the two factor graphs. For some invariants and graph products, there are challenging conjectures on the relations between the invariant of a product graph to the corresponding invariant of the factor graphs. In this thesis, we study a conjecture on the domination number of the Cartesian product of graphs, which was proposed by V. G. Vizing [2] in 1968. For graphs $G$ and $H$, denote by $G\Box H$ the Cartesian product of $G$ and $H$.

**Conjecture 1.** For any graph $G$ and $H$,

$$\gamma(G\Box H) \geq \gamma(G)\gamma(H).$$

This conjecture has attracted a lot of attention and there are many partial results. One important progress on this conjecture is that Clark and Suen [1] proved that $\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$ for any graphs $G$ and $H$. There are some special families of graphs for which the conjecture is verified. One result in this direction is obtained by Barcalkin and German [5]. In this thesis, we also prove a partial result concerning this conjecture. Namely if $G$ has $k$ disjoint complete subgraphs $G_1, G_2, \ldots G_k$ and $\gamma_G(\bigcup\limits_{i=1}^{k} G_i) = k$ or if $G$ has $k$ disjoint sets $S_1, S_2, .., S_k$ of $V(G)$ and $|D| + |B_D| \geq k$ for any $D \subseteq V(G)$ where $B_D = \{\ i\ |S_i \not\subseteq N[D], S_i \cap D = \emptyset\}$, then $\gamma(G\Box H) \geq k\gamma(H)$.

## 1.1 Basic notation

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, the *neighborhood* of $v$ is the vertex set $N(v) = \{u \mid u \text{ is adjacent to } v\}$. The *close neighborhood* of $v$ is the vertex set $N[v] = N(v) \cup \{v\}$. For $S \subseteq V(G)$, $N_G[S]$ is the subset of vertices that are in $S$ or adjacent to a vertex of $S$.

For $D \subseteq V(G)$, $D$ is a *dominating set* of $G$ if $V(G) \subseteq \bigcup\limits_{v\in D} N[v]$, and the the minimum cardinality of a dominating set of $G$ is called *domination number* of G denoted by $\gamma(G)$. In other words, that is, $\gamma(G) = \min\limits_{D\subseteq V(G)}\{|D| : V(G) \subseteq \bigcup\limits_{v\in D} N[v]\}$. Moreover, if the range become smaller such as a subgraph $S$ of $G$ replaces the graph $G$, then we will write

$\gamma_G(S) = \min\limits_{D \subseteq V(G)} \{|D| : V(S) \subseteq \bigcup\limits_{v \in D} N[v]\}$ for the the minimum cardinality of a dominating set of $S$ in $G$ as well .

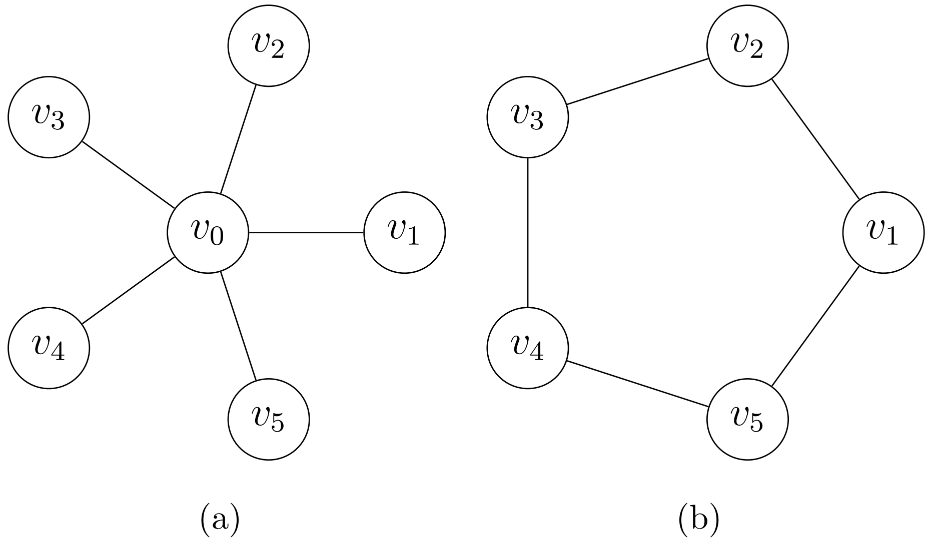



Figure 1.1: $K_{1,5}$ and $C_5$

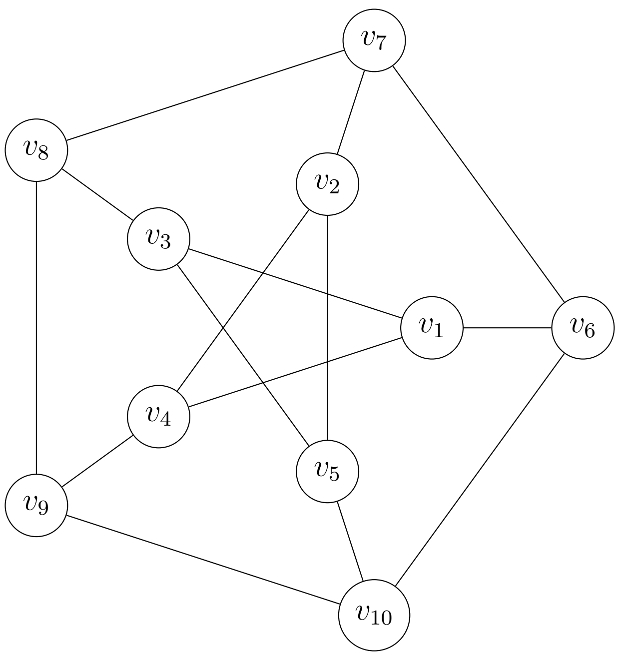



Figure 1.2: Petersen graph

For example, in Figure 1.1(a), $\{v_0\}$ is a dominating set of the graph $K_{1,5}$ and the domination number of $K_{1,5}$ is 1. In figure(2) $\{v_1, v_2, v_3\}$ and $\{v_1, v_4\}$ are dominating sets of the graph, and it can be checked that there is no dominating set with only 1 vertex. So the domination number of this graph is 2. In Figure 1.2, $S$ is a subgraph of $G$ with $S = G - \{v_4\} - \{v_8\} - \{v_9\}$. Since $V(S) \subseteq N[v_6] \bigcup N[v_5]$, $\gamma_G(S) = 2$.

A set $X \subseteq V(G)$ is called a 2-*packing* if $d(u, v) > 2$ for any different vertices $u$ and $v$ of $X$. The 2-*packing number*$\rho_2(G)$ is the maximum order of a 2-packing of $G$.

For two graphs $G$ and $H$, the *Cartesian product* $G \Box H$ is the graph with vertex set $\{(u, v)|u \in v(G), v \in V(H)\}$ and $(u, v)(u', v') \in E(G \Box H)$ if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

Figures on the below side show a example of $P_4 \Box P_4$ .

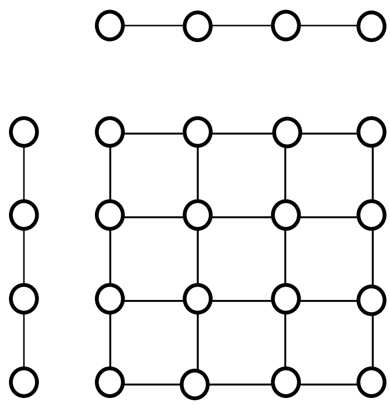


Figure 1.3: $P_4 \Box P_4$

## 1.2 History of Vizing conjecture

The concept of domination can be motivated by several examples. Consider for instance the problem of optimally placing fire stations in some towns so that every town either has a fire station or is a neighbor of a town which has a fire station. To save money, the county wants to build the minimum number of fire stations satisfying the above requirements

For the fire station problem, we consider the graph $G$ having all towns of the county as its vertices and a town is adjacent to its neighbor towns. The fire station problem is just the domination problem, as $\gamma(G)$ is the minimum number of fire stations needed. For the king domination problem on an chessboard, we consider the kings graph $G$, whose vertices correspond to the squares in the chessboard and two vertices are adjacent if and only if their corresponding squares have a common point or side. For the Chinese king domination problem, the vertex set is the same but two vertices are adjacent if and only if their corresponding squares have a common side.

In 1963, V.G.Vizing [2] conjectured that for any graph $G$ and $H$,

$$\gamma(G \square H) \geq \gamma(G)\gamma(H). \tag{1}$$

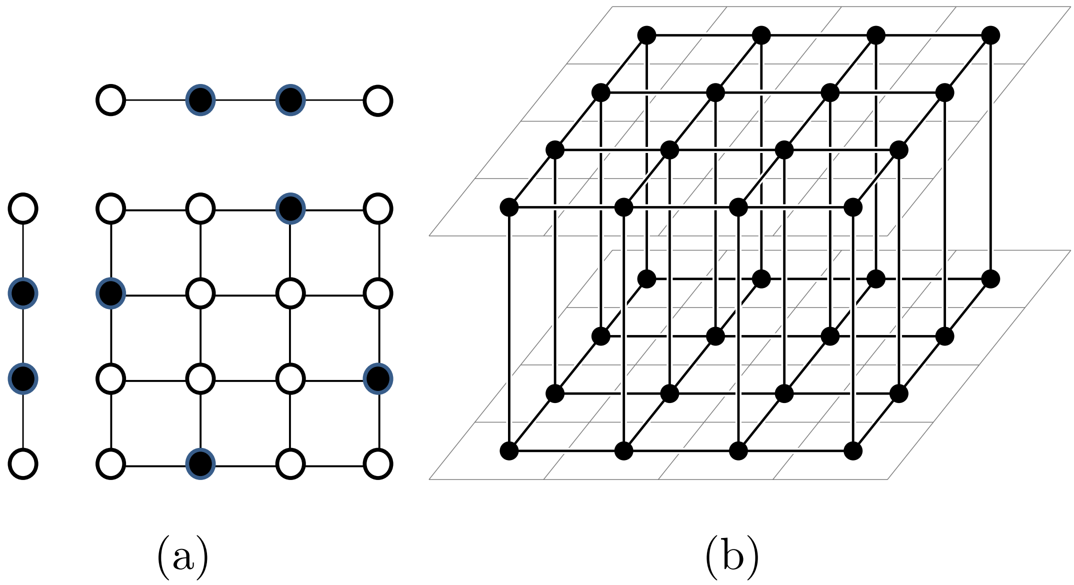



Figure 1.4: $P_4 \square P_4$ and $P_4 \square P_4 \square K_2$

Figure 1.4 (a) shows the domination number of $P_4 \square P_4$ equal to

$\gamma(P_4)\gamma(P_4)$, Figure 1.4 (b) shows $\gamma(P_4 \Box P_4 \Box K_2) = 7 > \gamma(P_4)\gamma(P_4)\gamma(K_2) = 2 \times 2 \times 1 = 4$.

Vizing conjecture is an active area of research spanning over forty years. Most of the research focus on proving the conjecture for certain classes of graphs. For example, in 1979, Barcalkin and German [5] proved that Vizing conjecture holds for graphs satisfying a certain "partitioning condition" on the vertex set. The idea of a "partitioning condition" inspired work for the next several decades, and Vizing conjecture was shown to hold on paths, trees, cycles, chordal graphs. Sun [2] showed that Vizing conjecture holds on graphs with $\gamma(G) \leq 3$. In 2000, Clark and Suen [1] showed that $2\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.

Vizing conjecture is easily stated, easily understood, however, its proof has eluded graph theorists for decades. These made the conjecture very attractive, and has generated a lot of research.

# Chapter 2

# Known Results

## 2.1 BG graphs

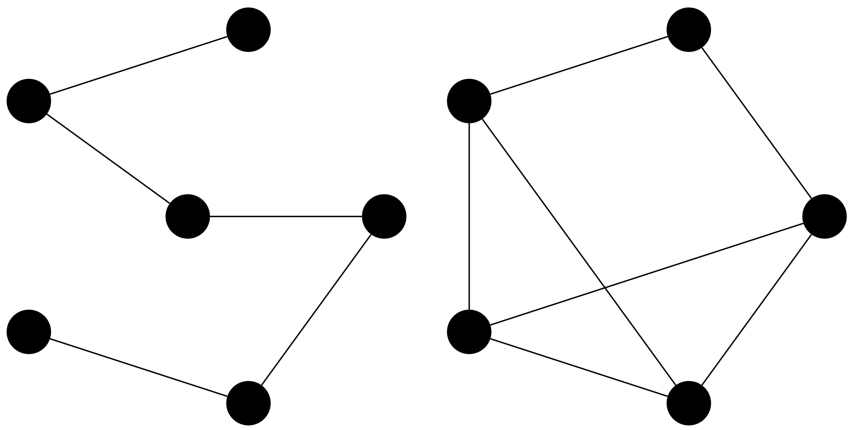
One idea used in proving special cases of the conjecture is to partition a graph into subgraphs of particular types. This was initiated by Barcalkin and German [7].

Let $G$ be a graph with domination number $k$. If the vertex set of $G$ can be covered by $k$ complete subgraphs, then $G$ is called a *decomposable graph*.

**Theorem 1. (Barcalkin and German [5])**
*If $G$ is a spanning subgraph of a decomposable graph $G'$ and $\gamma(G) = \gamma(G')$, then for every graph $H$,*

$$\gamma(G \Box H) \geq \gamma(G)\gamma(H).$$

A graph $G$ satisfies the hypothesis of Theorem 1 is called a *BG graph*. For example, trees, cycles, any graph with domination number 2 are BG graphs, and any graph having a 2-packing of cardinality equal to its domination number.



(a) (b)

Figure 2.1: BG-graphs

## 2.2 Weaker version of vizing conjecture

In the study of Vizings conjecture, the following natural question was asked. Is there a constant $c > 0$ such that

$$\gamma(G\Box H) \geq c\gamma(G)\gamma(H)?$$

In 2000, Clark and Suen [1] made a breakthrough and proved that this weaker version of Vizing conjecture is true.

**Theorem 2. (Clark and Suen [1] )**
*For any graphs $G$ and $H$,*

$$\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H)$$

The factor 1/2 comes from the double counting of the vertices of the minimum dominating set. In this thesis, we also use double counting technique to prove that Vizing conjecture is true for some special classes of graphs.

# Chapter 3

# Results of this thesis

In this thesis, we prove Vizing conjecture for two special classes of graphs.

**Theorem 3.** *If a simple graph $G$ has $k$ disjoint complete subgraphs $G_1, G_2, \ldots G_k$ such that $\gamma_G(\bigcup_{i=1}^{k} G_i) = k$, then for any simple graph $H$,*

$$\gamma(G \Box H) \geq k\ \gamma(H).$$

*Proof.* Let $S$ be a dominating set of $G \Box H$. We shall find a set $C$ such that $|S| \geq |C| \geq k\ \gamma(H)$.

For $h \in V(H)$, let

$$S_h = S \cap (V(G) \times \{h\}).$$

Let

$$C = \{(i, h) \mid V(G_i) \times \{h\} \subseteq N_{G \Box H}[S_h]\}.$$

and

$$C_i = \{\ h \in V(H) \mid (i, h) \in C\},$$

and

$$C_h = \{\ i \mid 1 \leq i \leq k, (i, h) \in C\}.$$

It follows from the definition that

$$|C| = \sum_{i=1}^{k} |C_i| = \sum_{h \in V(H)} |C_h|.$$

First, we will show that $C_i$ is a dominating set of $H$ for $i = 1, 2, \ldots, k$. Let $h \in V(H) - C_i$. We shall show that there exists $h' \in C_i$ such that $hh' \in E(H)$.

Since $h \notin C_i$, i.e. $(i, h) \notin C$, by definition of $C$, we have $V(C_i) \times \{h\} \nsubseteq N_{G\Box H}[S_h]$. Therefore there exists $g \in V(G_i)$, such that $(g, h) \notin N_{G\Box H}[S_h]$. On the other hand, as $S$ is a dominating set of $G\Box H$, $(g, h)$ has a neighbour $(g, h')$ from $S$. This neighbour must belong to $G_i \times \{h'\}$. Since $G_i$ is complete, $V(G_i) \times \{h'\}$ have to be completely dominated by $(g, h') \in S_{h'}$. This means $h' \in C_i$. Hence $C_i$ is a dominating set of graph $H$, and therefore $\gamma(H) \leq |C_i|$.

Therefore

$$k\gamma(H) = \sum_{i=1}^{k} \gamma(H) \leq \sum_{i=1}^{k} |C_i| = |C|. \tag{1}$$

Next we will show $|S_h| \geq |C_h|$ for each $h$. First, let $g_i \in V(G_i)$ for $i = 1, 2, \ldots .k$.

It is not hard to see that

$$\{\ g \mid (g, h) \in S_h\} \cup \{\ g_i \mid i \notin C_h\}$$

is a dominating set of $\bigcup_{i=1}^{k} G_i$ in $G$ with cardinality $|S_h| + (\ k - |C_h|\ )$.

Hence $k - (\ |C_h| - |S_h|\ ) \geq \gamma_G(\bigcup_{i=1}^{\gamma(G)} G_i) \ = \ k$. This implies that $|S_h| \geq |C_h|$. Therefore

$$|S| = \sum_{h \in V(H)} |S_h| \geq \sum_{h \in V(H)} |C_h| = |C|. \tag{2}$$

Combining (1) and (2), we have

$$|S| \geq |C| \geq k\ \gamma(H).$$

This completes the proof of the theorem. $\square$

Theorem 3 is a strengthening of Theorem 5. Every BG-graph satisfies the condition of Theorem 3 with $\gamma(G) = k$. However, there are graphs $G$ that are not not BG-graphs, and for which it follows from Theorem 3that they satisfy vizing's conjecture. Figure 3.1 are two such graphs.

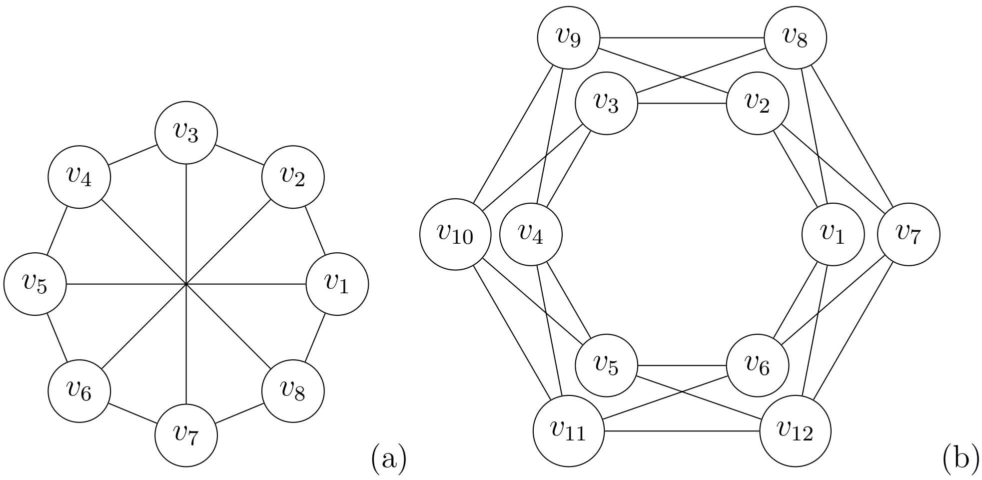



Figure 3.1: Non BG graphs

Let $G_1 = \{v_4, v_5\}$ $G_2 = \{v_1, v_2\}$ $G_3 = \{v_7, v_8\}$, then $\gamma_G(G_1 \bigcup G_2 \bigcup G_3) = 3$ in Figure 3.1 (a). For Figure 3.1 (b), let $G_1 = \{v_2, v_3\}$ $G_2 = \{v_7, v_8\}$ $G_3 = \{v_9, v_10\}$ $G_4 = \{v_11, v_12\}$, then $\gamma_G(G_1 \bigcup G_2 \bigcup G_3 \bigcup G_4) = 4$.

**Theorem 4.** *Let $S_1, S_2, .., S_k$ be pair-wise disjoint sets of $V(G)$. If for any $D \subseteq V(G)$, we have $|D| + |B_D| \geq k$ where $B_D = \{\ i\ |S_i \nsubseteq N[D], S_i \cap D = \emptyset\}$, then*

$$\gamma(G \Box H) \geq k\ \gamma(H)\ \textit{for any simple graph } H.$$

*Proof.* Let $D$ be a dominating set of $G \Box H$. For $h \in V(H)$, let $D_h = \{g | (g, h) \in D\}$. According to assumption, $|D_h| + |B_{D_h}| \geq k$ for each $h \in V(H)$.

It follows that we can send one vertex of $D_h$ to each $(S_i, h)$ where $i \notin B_{D_h}$ such that in this new disjoint sets $S_1(h), S_2(h), \ldots S_k(h)$ either $S_i(h) \cap D_h \neq \emptyset$ or $i \in B_{D_h}$ for any $i$ any $h$.

Let $D_i = (S_i(h) \times V(H)) \cap D$, $i = 1, 2, \ldots, k$.

Consider the projection of $D_i \times V(H)$ onto the graph $H$. That is

$$Pr_H(D_i \times V(H)) = \{h \in V(H)\ |S_i(h) \cap D \neq \emptyset\}.$$

We claim that $Pr_H(D_i \times V(H))$ is a dominating set of $H$ for $i = 1, 2, \ldots, k$.

Fix an index $i$. Suppose $h \in V(H) - Pr_H(D_i \times V(H))$. Then $i \in B_{D_h}$. So some vertices of $D$ in $(S_i, h')$ which did help $D_h$ to dominate $(S_i, h)$ completely, but they may have been removed in our new partition. As $(S_i, h') \cap D_{h'} \neq \emptyset$, we know that $i \notin B_{D_{h'}}$, it is the reason that we may have the other vertex of $D_{h'}$ to join $(S_i, h')$ such that $S_i(h') \cap D_{h'} \neq \emptyset$.

Therefore, $h$ is dominated by $Pr_H(D_i \times V(H))$. Hence $Pr_H(D_i \times V(H))$ is a dominating set of $H$, which follows $|D_i| \geq \gamma(H)$ for all $i$, and

$$|D| = \sum_{i=1}^{k} |D_i| \geq \sum_{i=1}^{k} \gamma(H) = k\ \gamma(H).$$

□

Now we wonder at the existence of the "$k$ " in Theorem 4 or what is the boundary of this "$k$ ". For these questions, we would like to recall 2-*packing number*$\rho_2(G)$.

**Recall.** A set $X \subseteq V(G)$ is called a 2-*packing* if $d(u, v) > 2$ for any different vertices $u$ and $v$ of $X$. The 2-*packing number*$\rho_2(G)$ is the maximum order of a 2-packing of $G$.

**Proposition 1.** *If $G$ has $k$ disjoint vertex sets $S_1, S_2, .., S_k$ such that any $D \subseteq V(G)$, $|D| + |B_D| \geq k$ where $B_D = \{\ i\ |S_i \not\subseteq N[D], S_i \cap D = \emptyset\}$ , then $\rho_2(G) \leq k \leq \gamma(G)$ for any graph $G$.*

*Proof.* First, assume that $k > \gamma(G)$ and $D$ is a dominating set of $G$, then we have $|D| + |B_D| = |D| + 0 \geq k > \gamma(G)$ that is a contradiction.

Next, for a 2-packing set $S$ of $G$, we name each vertex of $S$ by $S_1, S_2, \dots S_{\rho_2(G)}$, then $|D| + |B_D| \geq \rho_2(G)$ for any subset $D \subseteq V(G)$. $\square$

Recently Brešar and Rall [6] introduced the concept of a fair reception in a graph in the study of Vizing conjecture. Let $S_1, S_2, \dots, S_k$ be pair-wise disjoint sets of vertices of a graph $G = (V, E)$. Let $S = S_1 \cup S_2 \cup \cdots \cup S_k$ and let $Z = V - S$. We say that the sets $S_1, \dots, S_k$ form a *fair reception of size $k$* if for any $1 \leq l \leq k$ and any choice of $l$ sets $S_{i_1}, \dots, S_{i_l}$, the following holds: if $D$ externally dominates $S_{i_1}, \dots, S_{i_l}$ then

$$|D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) \geq l.$$

That is, on the left-hand side we count all the vertices of $D$ that are not in $S$, and for vertices of $D$ that are in some $S_j$ , we count all but one from $D \cap S_j$ .

**Theorem 5. (Brešar and Rall [6])** *If graph $G$ has a fair reception of size $k$, then*

$$\gamma(G \square H) \geq k \ \gamma(H).$$

Let $I$ be an independent set of vertices in the simple graph $G$. The least size of a set of vertices in $G$ that dominates I is $\gamma_G(I)$, and we denote the largest $\gamma_G(I)$ over all independent sets $I$ in $G$ by $\Gamma_I(G)$.

**Proposition 2.** *If a simple graph $G$ has a fair reception of size $k$, then $k \geq \Gamma_I(G)$.*

In the following, we show that the conditions of Theorem 5 and Theorem 4 are equivalent.

For any $D \subseteq V(G)$, let

$$\begin{aligned} A_D &= \{\ i\ |S_i \subseteq N[D], S_i \cap D = \emptyset\} \\ B_D &= \{\ i\ |S_i \not\subseteq N[D], S_i \cap D = \emptyset\} \\ C_D &= \{\ i\ |S_i \cap D \neq \emptyset\}. \end{aligned}$$

By definition, $|A_D| + |B_D| + |C_D| = k$. Therefore

$$\begin{aligned} & |D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} (|S_j \cap D| - 1) \geq l \\ \Leftrightarrow\ & |D \cap Z| + \sum_{j, S_j \cap D \neq \emptyset} |S_j \cap D| - \sum_{j, S_j \cap D \neq \emptyset} 1 \geq l \\ \Leftrightarrow\ & |D| - \sum_{j, S_j \cap D \neq \emptyset} 1 \geq l \\ \Leftrightarrow\ & |D| - |C_D| \geq |A_D| \\ \Leftrightarrow\ & |D| + |B_D| \geq k. \end{aligned}$$

It is quite natural to ask below question.

**Question 1.** For a simple graph $G$, can we always find $\gamma(G)$ disjoint sets of $V(G)$ to satisfy the condition of the Theorem 4 ? If this can be done, then vizing conjecture will be true.

**Definition 1.** *$G$ is called $\gamma_1$-$k$-critical graph if $k = \gamma(G) > \gamma(G + e)$ for any $e \in E(G^c)$, and $G$ is called $\gamma_2$-$k$-critical graph if $k = \gamma(G) < \gamma(G - e)$ for any $e \in E(G)$.*

**Theorem 6.** *If $G$ is a $\gamma_1$ -$k$-critical graph and it has $S_1, S_2, \ldots, S_k$ disjoint sets of $V(G)$ such that for any $D \subseteq V(G)$, we have $|D| + |B_D| \geq k$ , then $S_i$ is complete for all $i$.*

*Proof.* Assume $S_i$ is not complete for some $i$. Therefore, we can add an edge $e = v_1v_2$ to $S_i$. According to the definition of $\gamma_1$-k-critical graph , $\gamma(G) > \gamma(G + e) = k - 1$ and we can find a dominating set $D$ of the graph

$G - e$ with $|D| = k - 1$. It is easy to see this vertex set $D \subseteq V(G)$ can not keep $|D| + |B_D| \geq k$. Therefore, we have a contradiction.

□

Figure 3.2: $\gamma_1$-3-critical graphs

So the answer is "NO" to Question 1, it is not true for above graphs as they do not have 3 cliques to cover all vertices.

**Theorem 7.** *Let $D = \{v_1, v_2, \ldots v_k\}$ be a dominating set of a $\gamma_2$-k-critical graph G, then for any k disjoint complete subgraphs $G_1, G_2, \ldots G_k$ with $v_i \in V(G_i)$ for all i, $\gamma_G(\bigcup_{i=1}^{k} G_i) = k$.*

*Proof.* Let $G_1, G_2, \ldots G_k$ be such k disjoint complete subgraphs make sure each $v_i$ has been included in $V(G_i)$.

Assume $\gamma_G(\bigcup_{i=1}^{k} G_i) < k$, that is we can find a vertex set $S$ of $V(G)$ to dominate $\bigcup_{i=1}^{k} G_i$ with $|S| < k$. This means at least there are two vertices $v_i, v_j$ adjacent to a common vertex $s$ from $S$.

Now we remove $sv_i = e$ from $E(G)$, by the definition of $\gamma_2$-k-critical graph $\gamma(G) < \gamma(G - e)$, but $D = \{v_1, v_2, \ldots v_k\}$ is still a dominating set of $G - e$ this leads a contradiction.

□

We always can add some edges to a $\gamma_2$-k-critical graph to construct a $\gamma_1$-*k-critical graph.* So how about the converse side? Is it possible we can add edges to any $\gamma_1$-k-critical graph to construct a $\gamma_2$-(k-1)-critical graph ?

If this can be done, then every $\gamma_1$-k-critical graph $G'$ has $k-1$ disjoint complete subgraphs $G_1, G_2, \dots G_{k-1}$ such that $\gamma_{G'}(\bigcup_{i=1}^{k-1} G_i) = k-1$ since deleting edges from the $\gamma_2$-k-critical graph $G''$ will not reduce $\gamma_(\bigcup_{i=1}^{k-1} G_i) =$, this is, $\gamma_{G'}(\bigcup_{i=1}^{k-1} G_i) \geq \gamma_{G''}(\bigcup_{i=1}^{k-1} G_i)$ is always true for any $\bigcup_{i=1}^{k-1} G_i$.

This implies $\gamma(G \square H) \geq (\gamma(G) - 1)\gamma(H)$ as every graph $G$ with $\gamma(G) = k$ can be added edges to construct a $\gamma_1$-k-critical graph $G'$ and $\gamma(G \square H) \geq \gamma(G' \square H) \geq \gamma(G'' \square H) \geq (\gamma(G) - 1)\gamma(H)$.

However, this can not be made. The Figures in page 15 are examples to show how to develop a graph from $\gamma_1$-$V(G)$-critical graph to a $\gamma_1$-1-critical graph ,and you can not find a $\gamma_2$-2-critical graph between the $\gamma_1$-3-critical graph and the $\gamma_1$-2-critical graph.

A $\gamma_2$-k-critical graph is a basic condition for Theorem 3, but any $\gamma_1$-k-critical graph can be constructed by adding edges to the $\gamma_2$-k-critical graph may not satisfy the condition of Theorem 4. Because Theorem 4 is stronger than Theorem 3, between adding edges from the $\gamma_2$-k-critical graph to the $\gamma_1$-k-critical graph we can always have some examples (Fig (a) , (b) in P.11) that satisfy the condition of Theorem 4.

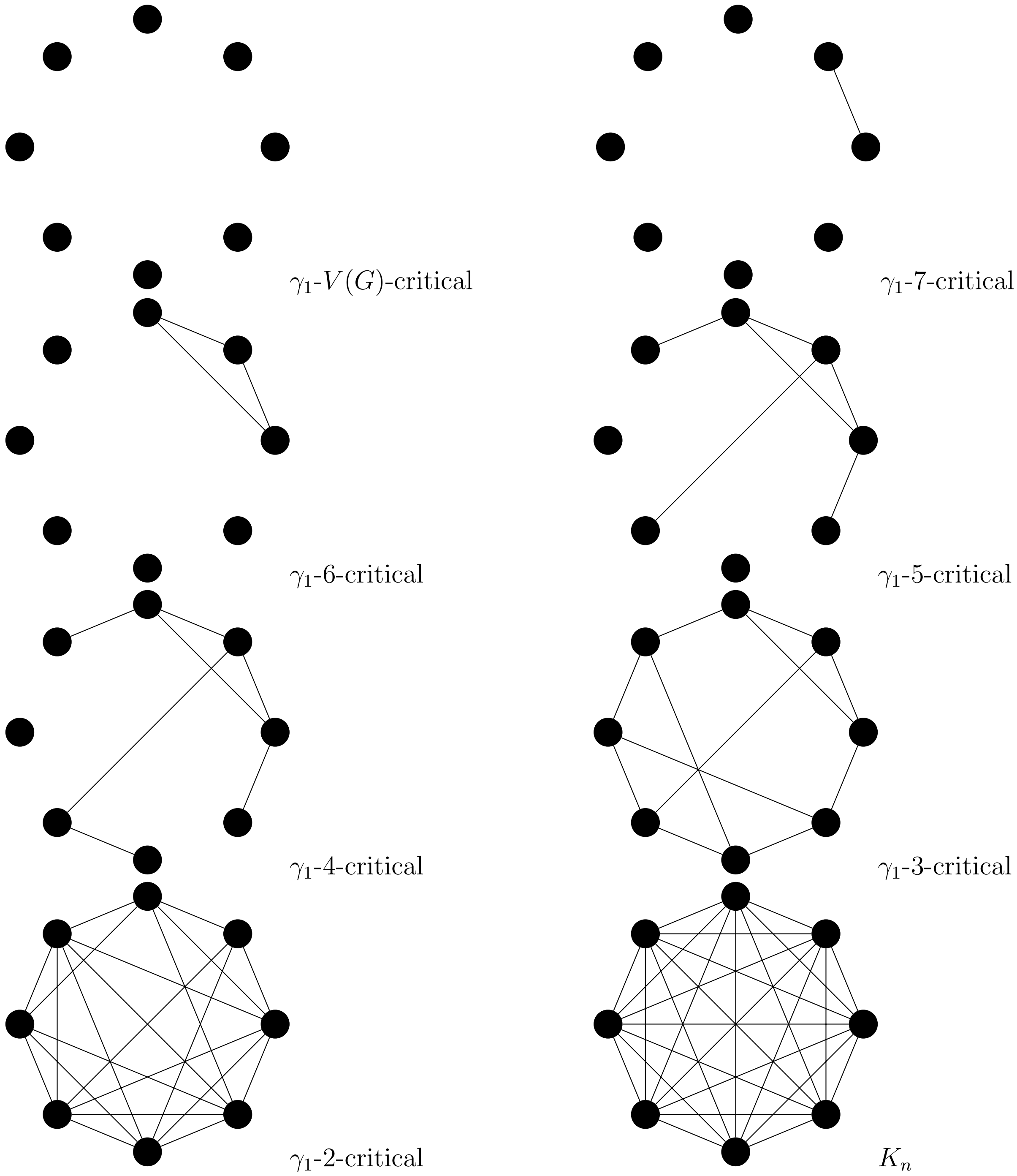



Figure 3.3: critical graphs

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