Numerical Computation for Nonlinear Beam Problems

by

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Abstract

Beam problem is very important for engineering theoretically and practically. In this thesis we study such kind of nonlinear 4-th order ordiniary differential equations with nonlinear boundary conditions. The well-posedness of these boundary value problems will be discussed. Moreover, we will design different schemes to solve them, through differential equation, integral equation or minimization. Each type can further be discretized by finite difference, finite element or spectral method, etc. In the end we will compare all methods and find the best one.

Contents

Chapter 1

Methods for Differential Equation

1.1 Introduction

Beam problem is very important for engineering theoretically and practically. It is used in construction, bridge, and machinery, etc. The governing equation is the classical Euler Bernoulli equation

$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{\rho A}\frac{\partial^4 y}{\partial x^4} = 0.$$

In this thesis we only consider its steady state solution, so it is a fourth order ODE. The normal linear boundary conditions are

(a) clamped: $y(p) = y'(p) = 0$,
(b) simply supported: $y(p) = y''(p) = 0$,
(c) free: $y''(p) = y'''(p) = 0$,
(d) sliding clamped: $y'(p) = y'''(p) = 0$.

Here, we plan to study such kind of nonlinear 4-th order ordiniary differential equation, coupled with nonlinear boundary conditions. The typical equation is as follows

$$u^{(4)}(x) - m\left(\int_0^L |u'(s)|^2 ds\right) u''(x) = f(x, u(x)), 0 < x < L, \tag{1.1}$$

Nonlinear boundary conditions of interest are
(e) resting on a bearing of elasticity: $y''(p) = 0,\ y'''(p) = -g(y(p))$,
(f) sliding on a bearing of elasticity: $y'(p) = 0,\ y'''(p) = -g(y(p))$,

where $p = 0$ or L. Some overviews can be found in Grossinho and Tersian [3], Ma [4, 10], and Grossinho and Ma [5]. These boundary value problems may not have solution or may have multiple solutions. Its well-posedness will be discussed in chapter 3.

The equation $(*)$ is a generalization of the stationary problem related to the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^2} - \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0.$$

It was proposed by Woinowsky-Frieger [1] as a model for the deflection of an extensible beam of length L with hinged ends. The quantity H represents an axial force, E the Young elasticity modulus, ρ the density, I the cross-sectional moment of inertia and A the cross-sectional area. The kind of correction in the brackets was early proposed by G. Kirchhoff to generalize the D'Alembert's equation with clamped ends. Therefore, the equation is often called Kirchhoff type beam equation. An interesting discussion on the various models for beam equations is presented in Arosio [2]. The first mathematical studies on Kirchhoff type beam equation were given in Dickey and Ball.

In this thesis, we try many numerical schemes to compute these type of boundary value problems. We solve the differential equations in chapter 1, by minimization techniques in chapter 2, and via integral equation in chapter 3. In the end we will compare all methods to find the best one.

In chapter 1, we discuss several models and compute their numerical solutions. We first discretize the problem by Finite Difference Method in section 1.2, then solve the nonlinear system by Mathematica build-in function NSolve, Newton's method, Fixed Point Method, JOR and SOR in each subsections, respectively. We employ the Weighted Residual Method in section 1.3.

1.2 Finite Difference Method

Consider the fourth order nonlinear equation

$$u^{(4)}(x) - m\left(\int_0^L |u'(s)|^2 ds\right) u''(x) = f(x, u(x)), \ 0 < x < L, \qquad (1.2)$$

subject to the nonlinear boundary conditions

$$u(0) = u''(0) = u''(1) = 0,$$

$$u^{(3)}(L) - m\left(\int_0^L |u'(s)|^2 ds\right) u'(L) = g(u(L)),$$

where $m \in C(\mathbf{R}^+)$, $f \in C([0, L] \times \mathbf{R})$ and $g \in C(\mathbf{R})$ are real functions.

Let $0 = x_0 < x_1 < \cdots < x_n = L$ be a partition of the interval $[0, L]$ with uniform mesh size $h = x_i - x_{i-1} = \frac{1}{n}$. Assume $u_i = u(x_i)$ and $f_i = f(x_i, u_i)$. Using central differences formula, the equation (1.1) becomes

$$\begin{gathered} u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2} - \widehat{m}(u) h^2 (u_{i+1} - 2u_i + u_{i-1}) = h^4 f_i \\ \Rightarrow u_{i-2} - \left(4 + \widehat{m}(u) h^2\right) u_{i-1} + \left(6 + 2\widehat{m}(u) h^2\right) u_i - \left(4 + \widehat{m}(u) h^2\right) u_{i+1} + \\ u_{i+2} = h^4 f_i, \ 2 \le i \le n-2, \end{gathered}$$

where $\widehat{m}(u)$ is a finite difference approximation of $m\left(\int_0^L u'^2 ds\right)$. In fact, from the trapezoidal formula we get

$$\int_0^L |u'(x)|^2 dx \approx \sum_{i=0}^{n-1} \frac{1}{2} h \left[u'(x_i)^2 + u'(x_{i+1})^2\right]$$

$$= \frac{1}{2} h \left[\left(u'(x_0)\right)^2 + \left(u'(x_n)\right)^2\right] + h \sum_{i=0}^{n-1} \left(u'(x_i)\right)^2$$

$$= \frac{1}{2} h \left(\frac{u_n - u_{n-1}}{h}\right)^2 + h \sum_{i=0}^{n-1} \left(\frac{u_{i+1} - u_{i-1}}{2h}\right)^2$$

$$= \frac{1}{2h} (u_n - u_{n-1})^2 + \frac{1}{4h} \sum_{i=0}^{n-1} (u_{i+1} - u_{i-1})^2,$$

and therefore $\widehat{m}(u)$ is now well-defined.

Taking into account the boundary conditions, we infer that

$$u(0) = u_0 = 0,$$

$$u'(0) = 0 \Rightarrow \frac{u_1 - u_{-1}}{2h} = 0 \Rightarrow u_1 = u_{-1},$$

$$u''(1) = 0 \Rightarrow \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = 0 \Rightarrow u_{n+1} = 2u_n - u_{n-1},$$

$$u^{(3)}(L) - m\left(\int_0^L |u'(s)|^2 ds\right) u'(L) = g(u(L))$$

$$\Rightarrow \frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h^3} - \widehat{m}(u)\frac{u_{n+1} - u_{n-1}}{2h} = g(u_n)$$

$$\Rightarrow u_{n+2} = 2h^3 g(u_n) + u_{n-2} - \left(4 + 2h^2\widehat{m}(u)\right)u_{n-1} + \left(4 + 2h^2\widehat{m}(u)\right)u_n.$$

Then we can compute $u_1, \cdots, u_n$ by solving the nonlinear system $A\mathbf{u} = \mathbf{b}$ with

$$A = \begin{bmatrix} b+1 & a & 1 & & & & & & \\ a & b & a & 1 & & & & & \\ 1 & a & b & a & 1 & & & & \\ & & & & \cdots & & & & \\ & & & & & 1 & a & b & a & 1 \\ & & & & & & 1 & a & b-1 & a+2 \\ & & & & & & & 2 & 2-b & 2a+2b-2 \end{bmatrix},$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} h^4 f_1 \\ h^4 f_2 \\ \vdots \\ \vdots \\ \vdots \\ h^4 f_{n-1} \\ h^4 f_n - 2h^3 g(u_n) \end{bmatrix},$$

where $a = -\left(4 + \widehat{m}\left(u\right)h^2\right)$ and $b = \left(6 + 2\widehat{m}\left(u\right)h^2\right)$.

In the practial computation we consider the typical model with

$$f\left(x\right) = -40x^3 + 48x^2 + 120x - 56,$$

$$m\left(s\right) = 1 + \frac{315}{353}s,$$

$$g\left(s\right) = 10s^3,$$

$$L = 1.$$

Its solution is

$$u\left(x\right) = x^5 - 2x^4 + 2x^2.$$

The computer used is P4 1.5GHz with 256MB memory. All the computation is ander the Mathematica 5 system.

The methods for solving the nonlinear system of equations $A\mathbf{u} = \mathbf{b}$ include Mathematica build-in function, Fixed Point method, JOR, SOR, and Newton's method etc. We employ each individual method in the following subsections.

1.2.1 Mathematica Solver

In this subsection, we use the built-in function NSolve in Mathematica to slove the nonlinear system directly. Its numerical results are given below.

n	Error $:= \max \lvert u_i - u(x_i) \rvert$
4	0.0356
5	0.0236
6	0.0166
7	0.0120
8	0.0094
9	0.0074
10	0.0060
11	0.0049
12	0.0041

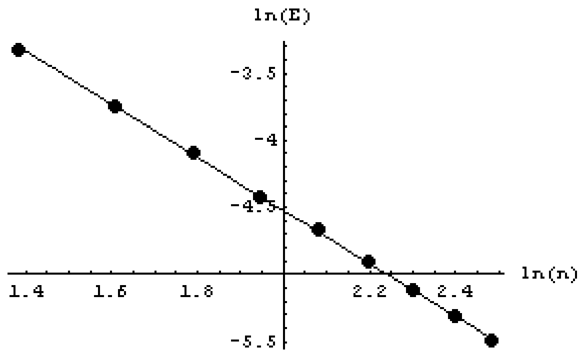

Figure 1.1

The asymptotic error formula is

$\ln E = -0.579983 - 1.97235 \ln(n)$

$\Rightarrow \ln E = \ln\left(\exp^{-0.579983} \times n^{-1.97235}\right)$

$\Rightarrow E = \frac{\exp^{-0.579983}}{n^{1.97235}} = \frac{0.367431}{n^{1.97235}} = 0.559908 \times h^{1.97235}$.

So $E \approx O\left(h^{2.0}\right) \to 0$, as $h \to 0$.

The disadvantage of this method is that solving the nonlinear equations is very expensive, especially when $n > 13$. The reason is that Mathematica produces many complex solutions, but only the unique real solution is needed. We discover that $n = \frac{1}{h}$ and numbers of solutions N have the relation with $N = 6n - 3$.

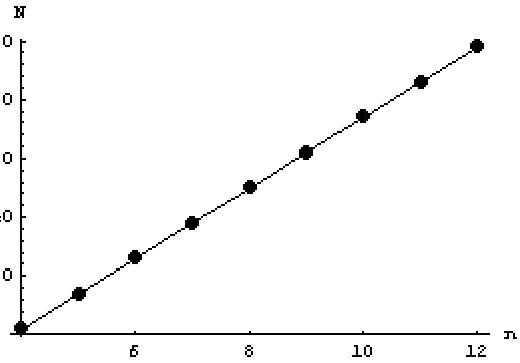

Figure 1.2

Although the method converges, computational costs is too high. Consequently, we should use other effective methods to solve the nonlinear system.

1.2.2 Fixed Point Method

Fixed point method is a common method to find solutions of nonlinear equations. But it may not always converge. In this subsection, we try to rectify the equation system to make it converge. For example,

$$\begin{cases} u_1 = \frac{1}{5}\left(\frac{h^4 f_1 - a u_2 - u_3}{b+1} + 4u_1\right) \\ u_2 = \frac{1}{5}\left(\frac{h^4 f_2 - a u_1 - a u_3 - u_4}{b} + 4u_2\right) \\ u_i = \frac{1}{2.5}\left(\frac{h^4 f_i - u_{i-2} - a u_{i-1} - a u_{i+1} - u_{i+2}}{b} + 1.5u_i\right), \ 3 \le i \le n-2 \\ u_{n-1} = \frac{1}{4}\left(\frac{h^4 f_{n-1} - (n-3)u_{n-3} - a u_{n-2} - (a+2)u_n}{b-1} + 3u_{n-1}\right) \\ u_n = \frac{1}{4}\left(\frac{h^4 f_n - 2h^3 g(u_n) - 2u_{n-2} - (2-b)u_{n-1}}{2a+2b-2} + 3u_n\right) \end{cases},$$

converges, while

$$\begin{cases} u_1 = \frac{1}{1.1}\left(\frac{h^4 f_1 - a u_2 - u_3}{b+1} + 0.1u_1\right) \\ u_2 = \frac{1}{1.1}\left(\frac{h^4 f_2 - a u_1 - a u_3 - u_4}{b} + 0.1u_2\right) \\ u_i = \frac{1}{1.1}\left(\frac{h^4 f_i - u_{i-2} - a u_{i-1} - a u_{i+1} - u_{i+2}}{b} + 0.1u_i\right), \ 3 \le i \le n-2 \\ u_{n-1} = \frac{1}{1.1}\left(\frac{h^4 f_{n-1} - (n-3)u_{n-3} - a u_{n-2} - (a+2)u_n}{b-1} + 0.1u_{n-1}\right) \\ u_n = \frac{1}{1.1}\left(\frac{h^4 f_n - 2h^3 g(u_n) - 2u_{n-2} - (2-b)u_{n-1}}{2a+2b-2} + 0.1u_n\right) \end{cases},$$

diverges, respectively, where $a = -\left(4 + \widehat{m}(u)\, h^2\right)$ and $b = \left(6 + 2\widehat{m}(u)\, h^2\right)$.

We use the previous approximate solution as an initial guess for the fine grids. Then its numerical results are given below. Let $\varepsilon = 10^{-6}$.

n	Iteration numbers	Error
4	342	0.0357448
5	686	0.0237507
6	1033	0.0168571
7	1449	0.0126628
8	1882	0.0100318
9	2266	0.0083931

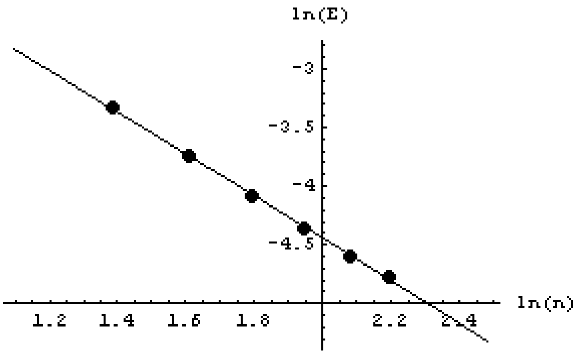

Figure 1.3

The asymptotic error formula is

$\ln E = -0.837648 - 1.80562 \ln n$

$\Rightarrow E = \frac{\exp^{-0.837648}}{n^{1.80562}} = 0.432727 \times h^{1.80562} \approx O\left(h^{1.8}\right) \to 0$, as $h \to 0$.

1.2.3 JOR and SOR Methods

In this subsection, we consider a linearization of the nonlinear systems and then apply the classical JOR and SOR iterative methods to solve the linear system.

JOR Method

In every iteration, substituting the approximation solution into the matrix A and linearize the system.

- Algorithm:

  Choose initial $u^{(0)}$. Repeat:

  1. Decompose

$$A(\mathbf{u}^{(k)}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ a_{21} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn-1} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1n} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$= D + L + U.$$

  2. JOR iteration: $\mathbf{u}^{(k+1)} = \left(I - wD^{-1}A\right)\mathbf{u}^{(k)} + wD^{-1}\mathbf{b}$.

Remark 1 *JOR converges* $\Leftrightarrow \rho\left(I - wD^{-1}A\right) < 1$.

The entry is $\max\left[\rho\left(I - wD^{-1}A\right)\right]$ in the table, $u^{(0)}$ is initial approximate value, and $u^{(n)}$ is final approximate value.

- $u^{(0)}$

w	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
0.1	0.99823	0.99927	0.99965	0.99981	0.99989	0.99993
0.3	0.99469	0.99782	0.99873	0.99945	0.99968	0.99980
0.5	0.99115	0.99637	0.99828	0.99909	0.99947	0.99967
0.7	0.98761	0.99492	0.99760	0.99873	0.99926	0.99955
0.9	1.22068	1.30098	1.34314	1.36698	1.38147	1.39078
1.1	1.71416	1.81231	1.86384	1.89298	1.91069	1.92206

- $u^{(n)}$

w	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
0.1	0.99818	0.99926	0.99965	0.99981	0.99989	0.99993
0.3	0.99455	0.99780	0.99872	0.99945	0.99968	0.99980
0.5	0.99092	0.99634	0.99827	0.99909	0.99947	0.99967
0.7	0.98729	0.99488	0.99759	0.99872	0.99926	0.99954
0.9	1.21807	1.30053	1.34292	1.36686	1.38140	1.39072
1.1	1.71097	1.81175	1.86357	1.89283	1.91060	1.92200

Choose the best w to be 0.7.

We use previous approximate solution as initial approximation for finer grid. Then the numerical results are given below. Let $\varepsilon = 10^{-5}$.

n	Iteration numbers	Error
4	79	0.0397
5	211	0.0247
6	322	0.0181
7	427	0.0144
8	490	0.0126
9	483	0.0124

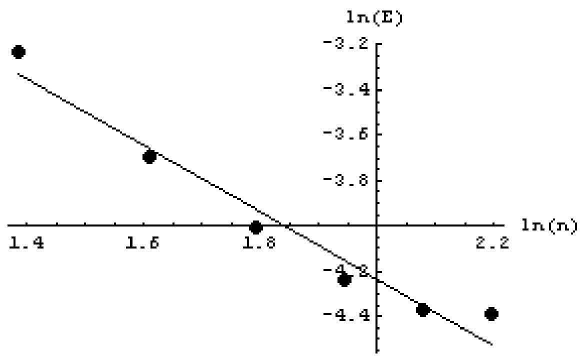

Figure 1.4

The asymptotic error formula is

$\ln E = -1.29346 - 1.47007 \ln n$

$\Rightarrow E = \frac{\exp^{-1.29346}}{n^{1.47007}} = \frac{0.27432}{n^{1.47007}} = 0.27432 \times h^{1.47007} \approx O\left(h^{1.5}\right) \to 0,$ as $h \to 0$.

SOR Method

- Algorithm :

  Choose initial $u^{(0)}$. Repeat:

  1. Decompose $A(u^{(k)}) = D + L + U$ as in JOR.

  2. SOR iteration

  $u^{(k+1)} = (D + wL)^{-1} [(1 - w) D - wU] u^{(k)} + w (D + wL)^{-1} B.$

Remark 2 *SOR converges* $\Leftrightarrow \rho\left((D + wL)^{-1} [(1 - w) D - wU]\right) < 1.$

The entry is $\max\left[\rho\left((D + wL)^{-1} [(1 - w) D - wU]\right)\right]$ in the table, $u^{(0)}$ is initial approximate value, and $u^{(n)}$ is final approximate value.

- $u^{(0)}$

w	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
0.1	0.99813	0.99923	0.99963	0.99980	0.99989	0.99993
0.3	0.99377	0.99743	0.99879	0.99936	0.99963	0.99977
0.5	0.98825	0.99516	0.99771	0.99879	0.99930	0.99957
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.5	0.88024	0.95342	0.97853	0.98879	0.99359	0.99608
1.7	0.81708	0.86582	0.95080	0.97634	0.98696	0.99219
1.9	1.14463	1.15417	1.18397	1.22087	1.25632	1.27493

- $u^{(n)}$

w	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
0.1=	0.99809	0.99923	0.99963	0.99980	0.99989	0.99993
0.3	0.99361	0.99742	0.99878	0.99935	0.99963	0.99977
0.5	0.98795	0.99513	0.99770	0.99878	0.99930	0.99957
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1.5	0.87678	0.95317	0.97843	0.98875	0.99358	0.99607
1.7	0.81613	0.86564	0.95052	0.97625	0.98693	0.99217
1.9	1.14443	1.15401	1.18345	1.22076	1.25626	1.27490

We find that the best w is 1.7.

We use previous approximate solution as initial approximation for finer grids. Then the numerical results are given below. Let $\varepsilon = 10^{-5}$.

n	Iteration numbers	Error
4	56	0.0396
5	45	0.0242
6	47	0.0171
7	50	0.0128
8	68	0.0097
9	58	0.0074

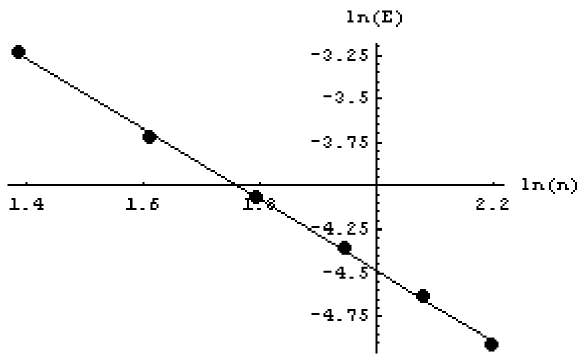

Figure 1.5

The asymptotic error formula is

$\ln E = -0.424811 - 2.03181 \ln n$

$\Rightarrow E = \frac{\exp^{-0.424811}}{n^{2.03181}} = 0.653893 \times h^{2.03181} \approx O\left(h^{2.0}\right) \to 0$, as $h \to 0$.

Obviously we can find that SOR's iteration number is less than that of JOR's under the same accuracy requirement. So SOR is superior JOR.

1.2.4 Newton's Method

In this subsection, we use Newton's method to solve the nonlinear system (1.2) $F(u_1, u_2, \ldots, u_n) = (F_1, \ldots, F_n)$. Recall that

$$\begin{array}{l} F_1 = (b+1)\,u_1 + au_2 + u_3 - h^4 f_1 \\ F_2 = au_1 + bu_2 + au_3 + u_4 - h^4 f_2 \\ F_i = u_{i-2} + au_{i-1} \cdots + u_{i+2} - h^4 f_i,\ 3 \le i \le n-2\ , \\ F_{n-1} = u_{n-3} + \cdots + (a+2)\,u_n - h^4 f_{n-1} \\ F_n = 2u_{n-2} + \cdots - h^4 f_n + 2h^3 g(u_n) \end{array} \tag{1.3}$$

where $a = -\left(4 + \widehat{m}(u)\,h^2\right)$ and $b = \left(6 + 2\widehat{m}(u)\,h^2\right)$.

- Newton's Algorithm :

1. Choose initial $u^{(0)}$.
2. Repeat until converge:

Compute the Jacobian $J(u_1, u_2, \ldots, u_n) = \left[\ \frac{\partial F}{\partial u_i}\ \right]$, and
$u^{(k)} = u^{(k-1)} - J\left(u_1^{(k-1)}, u_2^{(k-1)}, \ldots, u_n^{(k-1)}\right)^{-1} F\left(u_1^{(k-1)}, u_2^{(k-1)}, \cdots, u_n^{(k-1)}\right)$.

The adaptive method uses two error related to the Newton's iteration and step size h respectively. We use the approximate solution as an initial approximation for the fine grids. Using Newton's method to solve (1.3) gives the results below.

Adaptive Method

n	Iteration numbers	Error
2^2	2	0.0357224
2^3	2	0.0094351
2^4	2	0.0023607
2^5	2	0.0005903
2^6	2	0.0001475

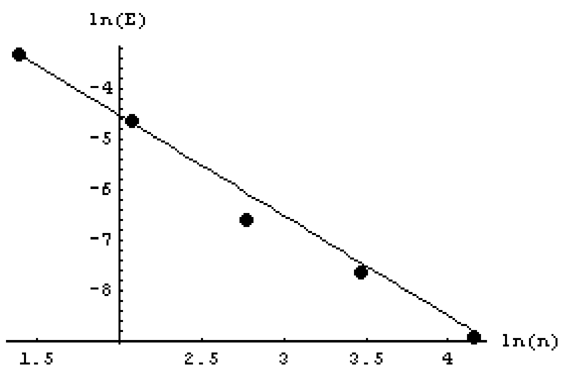

Figure 1.6

The asymptotic error formula is

$\ln E = -0.560753 - 1.98359 \ln n$

$\Rightarrow E = \frac{\exp^{-0.560753}}{n^{1.98359}} = 0.570779 \times h^{1.98359} \approx O\left(h^{2.0}\right) \to 0$, as $h \to 0$.

Newton's method takes less time than that solved by the built-in function of Mathematica as $n > 10$. Newton's method can solve even when $n = 60$, which costs only 20 minutes. But it is still time comsuming for large n since each iteration needs a evaluation of Jacobi matrix and solving the $n \times n$ system.

Therefore Newton's method is faster than the other methods in the previous of the subsections.

Now, we deal with the second testing problem

$$u^{(4)}(x) - M\left(\int_0^L |u'(x)|^2 dx\right) u''(x) = f(x, u(x)), \ 0 < x < L,$$

$$u(0) = u''(0) = u(L) = 0,$$

$$u''(L) = g(u'(L)).$$

Similarly, we use the finite differences method to solve the differential equation, and Newton's method to solve the discretized nonlinear equations. We consider $0 = x_0 < x_1 < \cdots < x_n = L$ be a discretization of the interval $[0, L]$ with mesh size $h_i = x_i - x_{i-1}$, $u_i = u(x_i)$ and $f_i = f(x_i, u_i)$, $\widehat{m}(u)$ is a finite difference approximation of $m\left(\int_0^L |u'(s)|^2 ds\right)$,

where $\int_0^L |u'(x)|^2 dx \approx \sum_{i=0}^{n-1} \frac{1}{2} h \left[u'(x_i)^2 + u'(x_{i+1})^2\right]$

$= \frac{1}{2} h \left(u'(x_0)\right)^2 + \left(u'(x_n)\right)^2 + h \sum_{i=0}^{n-1} \left(u'(x_i)\right)^2$

$= \frac{1}{2} h \left(\frac{u_n - u_{n-1}}{h}\right)^2 + h \sum_{i=0}^{n-1} \left(\frac{u_{i+1} - u_{i-1}}{2h}\right)^2 = \frac{1}{2h}\left(u_n - u_{n-1}\right)^2 + \frac{1}{4h} \sum_{i=0}^{n-1} \left(u_{i+1} - u_{i-1}\right)^2.$

Using the central differences formula, the differential equation is discretized as

- $u^{(4)}(x) - m\left(\int_0^L |u'(s)|^2 ds\right) u''(x) = f(x, u(x))$

$\Rightarrow u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2} - \widehat{m}(u) h^2 \left(u_{i+1} - 2u_i + u_{i-1}\right) = h^4 f_i$

$\Rightarrow u_{i-2} - \left(4 + \widehat{m}(u) h^2\right) u_{i-1} + \left(6 + 2\widehat{m}(u) h^2\right) u_i - \left(4 + \widehat{m}(u) h^2\right) u_{i+1} + u_{i+2} = h^4 f_i$, $2 \le i \le n - 2$.

Boundary conditions are approximated as follows

- $u(0) = u_0 = 0.$
- $u(1) = u_n = 0.$
- $u''(0) = 0 \Rightarrow u_1 = 2u_0 - u_{-1} = -u_{-1}.$
- $u''(1) = g(u'(1))$

$\Rightarrow \frac{u_{n+1}-2u_n+u_{n-1}}{h^2} = g\left(\frac{u_{n+1}-u_{n-1}}{2h}\right) = -2\left(\frac{u_{n+1}-u_{n-1}}{2h}\right)$
$\Rightarrow (h+1)\,u_{n+1} = 2u_n + (h-1)\,u_{n-1}.$

So it is reduced to a nonlinear system $A\mathbf{u} = \mathbf{b}$, with

$$A = \begin{bmatrix} b-1 & a & 1 & & & & & & \\ a & b & a & 1 & & & & & \\ 1 & a & b & a & 1 & & & & \\ & & & & \cdots & & & & \\ & & & & & 1 & a & b & a & 1 \\ & & & & & & & 1+h & a(1+h) & (h-1)+b(h+1) \\ & & & & & & & & & \end{bmatrix},$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \vdots \\ u_{n-1} \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} h^4 f_1 \\ h^4 f_2 \\ \vdots \\ \vdots \\ \vdots \\ h^4 f_{n-1} \end{bmatrix},$$

where $a = -\left(4 + \widehat{m}(u)\,h^2\right)$ and $b = \left(6 + 2\widehat{m}(u)\,h^2\right).$

We use previous approximate solution as an initial of Newton's method for fine grids. Then the numerical results are given below.

n	Iteration numbers	Error
10	2	0.00490
20	2	0.00199
30	2	0.00113
40	1	0.00076
50	2	0.00057
60	2	0.00045

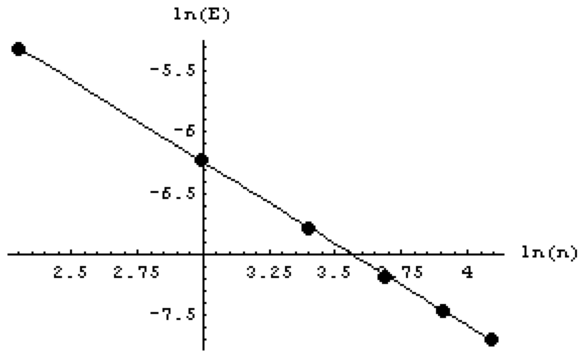

Figure 1.7

The asymptotic error formula is

$\ln E = -2.22393 - 1.34046 \ln(n)$

$\Rightarrow E = \frac{\exp^{-2.22393}}{n^{1.34046}} = 0.108183 \times h^{1.34046} \approx O\left(h^{1.3}\right) \to 0$, as $h \to 0$.

Next, we deal with the third testing problem

$$u^{(4)}(x) - M\left(\int_0^L |u'(x)|^2 dx\right) u''(x) = f(x, u(x)), \; 0 < x < L,$$

$$u''(0) = u''(L) = 0,$$

$$\begin{aligned} u^{(3)}(0) - m\left(\int_0^L |u'(s)|^2 ds\right) u'(0) &= g(u(0)), \\ u^{(3)}(L) - m\left(\int_0^L |u'(s)|^2 ds\right) u'(L) &= h(u(L)). \end{aligned}$$

Again we use the finite differences method to solve the differential equation, and Newton's method to solve the discretized nonlinear equations. We consider $0 = x_0 < x_1 < \cdots < x_n = L$ be a discretization of the interval $[0, L]$ with mesh size $h_i = x_i - x_{i-1}$, $u_i = u(x_i)$ and $f_i = f(x_i, u_i)$, $\widehat{m}(u)$ is a finite difference approximation of $m\left(\int_0^L |u'(s)|^2 ds\right)$,

where $\int_0^L |u'(x)|^2 dx \approx \sum_{i=0}^{n-1} \frac{1}{2} h \left[u'(x_i)^2 + u'(x_{i+1})^2\right]$

$= \frac{1}{2}h\left[(u'(x_0))^2 + (u'(x_n))^2\right] + h\sum\limits_{i=0}^{n-1}(u'(x_i))^2$

$= \frac{1}{2}h\left(\frac{u_n - u_{n-1}}{h}\right)^2 + h\sum\limits_{i=0}^{n-1}\left(\frac{u_{i+1} - u_{i-1}}{2h}\right)^2 = \frac{1}{2h}(u_n - u_{n-1})^2 + \frac{1}{4h}\sum\limits_{i=0}^{n-1}(u_{i+1} - u_{i-1})^2.$

Using the central differences formula, the differential equation is discretized as

- $u^{(4)}(x) - m\left(\int_0^L |u'(s)|^2 ds\right)u''(x) = f(x, u(x))$

$\Rightarrow u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2} - \widehat{m}(u)\, h^2 (u_{i+1} - 2u_i + u_{i-1}) = h^4 f_i$

$\Rightarrow u_{i-2} - (4 + \widehat{m}(u)\, h^2)\, u_{i-1} + (6 + 2\widehat{m}(u)\, h^2)\, u_i - (4 + \widehat{m}(u)\, h^2)\, u_{i+1} + u_{i+2} = h^4 f_i$, $2 \leq i \leq n-2$.

Boundary conditions are approximated as follows

- $u''(0) = 0 \Rightarrow u_1 = 2u_0 - u_{-1}$.
- $u''(1) = 0 \Rightarrow u_{n+1} = 2u_n - u_{n-1}$.
- $u_{n+2} = 2h^3 h(u_n) + u_{n-2} - (4 + 2h^2\widehat{m}(u))\, u_{n-1} + (4 + 2h^2\widehat{m}(u))\, u_n$.
- $u'''(0) - m\left(\int_0^L |u'(x)|^2 dx\right)u'(0) = g(u(0))$

$\Rightarrow \frac{u_2 - 2u_1 + 2u_{-1} - u_{-2}}{2h^3} - \widehat{m}(u)\,\frac{u_1 - u_{-1}}{2h} = g(u_0)$

$\Rightarrow u_2 - 2u_1 + 2u_{-1} - u_{-2} - h^2\widehat{m}(u)(u_1 - u_{-1}) = 2h^3 g(u_0)$

$\Rightarrow u_2 - 2u_1 + 2(2u_0 - u_1) - u_{-2} - h^2\widehat{m}(u)(u_1 - (2u_0 - u_1)) = 2h^3 g(u_0)$

$\Rightarrow u_{-2} = -2h^3 g(u_0) + u_2 - 4u_1 + 4u_0 - h^2\widehat{m}(u)(2u_1 - 2u_0)$

$\Rightarrow u_{-2} = -2h^3 g(u_0) + u_2 - u_1(4 + 2h^2\widehat{m}(u)) + u_0(4 + 2h^2\widehat{m}(u))$.

The nonlinear system is $A\mathbf{u} = \mathbf{b}$, with

$$A = \begin{bmatrix} 2a+2b-2 & 2-b & 2 & & & & & & & \\ a+2 & b-1 & a & 1 & & & & & & \\ 1 & a & b & a & 1 & & & & & \\ & & & & \cdots & & & & & \\ & & & & & 1 & a & b & a & 1 \\ & & & & & & 1 & a & b-1 & a+2 \\ & & & & & & & 2 & 2-b & 2a+2b-2 \end{bmatrix},$$

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} h^4 f_1 \\ h^4 f_2 \\ \vdots \\ \vdots \\ \vdots \\ h^4 f_{n-1} \\ h^4 f_n - 2h^3 h(u_n) \end{bmatrix},$$

where $a = -\left(4 + \widehat{m}(u)\, h^2\right)$ and $b = \left(6 + 2\widehat{m}(u)\, h^2\right)$.

We use previous approximate solution as an initial of Newton's method for fine grids. Then the numerical results are given below.

n	Iteration numbers	Error
10	2	0.0619
20	2	0.0202
30	2	0.0111
40	2	0.0074
50	2	0.0055
60	2	0.0043

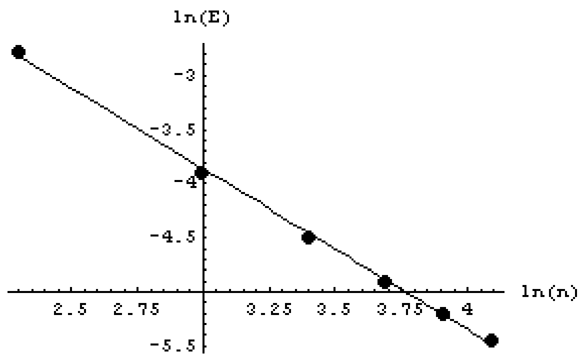

Figure 1.8

The asymptotic error formula is

$\ln E = 0.594566 - 1.48621 \ln(n)$

$\Rightarrow E = \frac{\exp^{0.594566}}{n^{1.48621}} = 1.81224 \times h^{1.48621} \approx O\left(h^{1.3}\right) \to 0$, as $h \to 0$.

1.3 Weighted Residual Method

In this subsection, we apply the weighted residual method for the beam problem $Lu = f$, where

$$Lu = u^{(4)}(x) - m\left(\int_0^L |u'(s)|^2 ds\right) u''(x).$$

First, choose $\{\phi_k(x)\}$ and $\{\Phi_i(x)\}$ as trial and test functions respectively. The basis function $\phi_k(x)$ may or may not all satisfy boundary conditions. Consider the numerical solution to be $\widehat{u}(x) = \sum_{k=0}^{n} c_k \phi_k(x)$. Weighted residual method find the coefficients c_k by the Galerkin condition

$$\int_0^L (L(\widehat{u}) - f)\Phi_i dx = 0 \Leftrightarrow \int_0^L L(\widehat{u})\Phi_i dx = \int_0^L f\Phi_i dx, \ \forall i = 1, \ldots, n.$$

Note that the Galerkin method uses $\{\phi_k(x)\} = \{\Phi_i(x)\}$.

Substituting $\widehat{u}(x) = \sum_{k=0}^{n} c_k \phi_k(x)$ into $\int_0^L L(\widehat{u})\Phi_i dx = \int_0^L f\Phi_i dx$, we have

$$\int_0^L \left(\sum_{k=0}^{n} c_k \phi_k^{(4)}(x)\right) \cdot \Phi_i dx - \int_0^L m\left(\int_0^L \left(\sum_{k=0}^{n} c_k \phi_k'(s)\right)^2 ds\right)\left(\sum_{k=0}^{n} c_k \phi_k''(x)\right) \cdot \Phi_i dx$$
$$- \int_0^L f\left(x, \sum_{k=0}^{n} c_k \phi_k(x)\right) \cdot \Phi_i dx = 0, \ \forall i = 1, \cdots, n.$$

Taking $\widehat{u}(x) = \sum_{k=0}^{n} c_k \phi_k(x)$ into the boundary conditions, we obtain

$$\sum_{k=0}^{n} c_k \phi_k(0) = \sum_{k=0}^{n} c_k \phi_k''(0) = \sum_{k=0}^{n} c_k \phi_k''(1) = 0,$$
$$\sum_{k=0}^{n} c_k \phi_k^{(3)}(x) - m\left(\int_0^L \left(\sum_{k=0}^{n} c_k \phi_k'(s)\right)^2 ds\right) \sum_{k=0}^{n} c_k \phi_k'(L) = g\left(\sum_{k=0}^{n} c_k \phi_k(L)\right).$$

Then it is reduced to be a nonlinear system for c_k.

Let choose trial function

$$\{\phi_k(x)\} = \{\sin(k\pi x), \cos(k\pi x)\}_{k=0}^{n},$$

and test function

$$\{\Phi_i\} = \{\sin(i\pi x)\}_{i=1}^{2n-3}.$$

So the numerical solution

$$\widehat{u}(x)=\sum_{k=1}^{n}c_k\sin(k\pi x)+\sum_{k=0}^{n}c_{n+1+k}\cos(k\pi x).$$

The numerical solutions of the corresponding nonlinear system has real and complex, and we only need to pick the real solutions. We first use the built-in function NSolve in Mathematica to slove the nonlinear system directly. The errors are listed below, where $n^{'}=2\times n+1$.

$n^{'}$	Error
7	0.65375
9	0.38611
11	0.17039
13	0.06592
15	0.02389

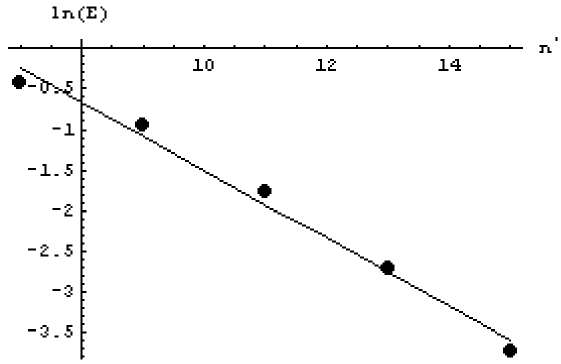

Figure 1.9

The asymptotic error formula is

$\ln E=2.69238-0.419303n^{'}$

$\Rightarrow E=\left(e^{2.69238}\right)\left(e^{-0.419303}\right)^{n^{'}}=14.7668\times 0.657505^{n^{'}}.$

It is time consuming when $n>8$.

The basis $\{\phi_k\}$ and $\{\Phi_i\}$ must be chosen properly to make it converge. For example, $\{\phi_k(x)\}=\{\sin(k\pi x)\}$ and $\{\Phi_i\}=\{\sin(i\pi x)\}$ are bad basis. The reason is that at $x=1$, $\widehat{u}(1)=\sum_{k=0}^{n}c_k\phi_k(1)=0$, but $u(1)=x^5-2x^4+2x^2|_{x=1}=1$.

We collect the convergent and divergent trial and test basis as below.

- Convergence:

$\{\phi_k(x)\}$	$\{\Phi_i\}$
$\{x^k\}$	arbitrarily
$\{\sin(k\pi x), \cos(k\pi x)\}$	$\{x^i\}$
	$\{\sin(i\pi x)\}$
$\{\sin(k\frac{\pi}{2}x), \cos(k\frac{\pi}{2}x)\}$	$\{\sin(i\pi x)\}$
	$\{\cos(i\frac{\pi}{2}x)\}$
	$\{\sin(i\frac{\pi}{2}x), \cos(i\frac{\pi}{2}x)\}$
	$\{x^i\}$
$\{\sin(k\frac{\pi}{2}x), x^1, x^2\}$	$\{\sin(i\frac{\pi}{2}x), x^1, x^2\}$
$\{\cos(k\frac{\pi}{2}x), x^1\}$	$\{\cos(i\frac{\pi}{2}x), x^1\}$

- Divergence:

$\{\phi_k(x)\}$	$\{\Phi_i\}$
$\{\sin(k\pi x), \cos(k\pi x)\}$	$\{\sin(i\pi x), \cos(i\pi x)\}$
$\{\sin(k\frac{\pi}{2}x), \cos(k\frac{\pi}{2}x)\}$	$\{\sin(i\frac{\pi}{2}x)\}$
$\{\sin(k\frac{\pi}{2}x)\}$	$\{\sin(i\frac{\pi}{2}x)\}$
$\{\cos(k\frac{\pi}{2}x)\}$	$\{\cos(i\frac{\pi}{2}x)\}$

Although we find several convergent basis for the equation, it is more important to construct faster convergent basis which will cost less time. Unfortunately, the relation between basis and its convergence is still an open problem.

Next we use Newton's method to solve the nonlinear system. Its results are as follows.

n'	Iteration numbers	Error
7	6	0.653753
9	6	0.386111
11	8	0.170397
13	10	0.065921
15	12	0.023891
17	14	0.008236
19	16	0.002751
21	18	0.000897

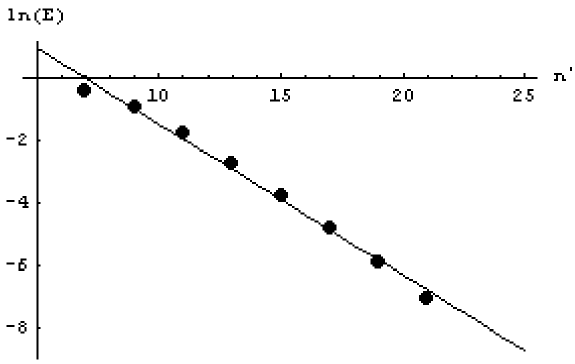

Figure 1.10

The asymptotic error formula is

$\ln E = 3.33314 - 0.481932^{n'}$

$\Rightarrow E = \left(e^{3.33314}\right)\left(e^{-0.481932}\right)^{n'} = 28.0262 \times 0.617589^{n'}$,

where $n' = 2 \times n + 1$.

Comparing all these existing methods stated above, we conclude that the weighted residual method combined with Newton's iteration to solve nonlinear system is the fastest method.

Chapter 2

Minimization Method

2.1 Introduction

In this chapter we study the model problem (1.2) with boundary conditions. It is equivalent to an optimization problem, which we will stat below. We begin with some notations. Let $H^k(0,L)$ be the Sobolev space of the functions $u : [0,L] \to R$ with the derivative $u^{(k-1)}$ absolutely continuous and $u^{(k)} \in L^2(0,L)$. we consider the solution on the Hilbert space

$$E = \left\{ u \in H^2(0,L) : u(0) = u'(0) = 0 \right\},$$

equiped with the inner product and norm

$$\langle u, v \rangle = \textstyle\int_0^L u''(x)\, v''(x)\, dx,\ ||u||_E = ||u''||_{L^2},$$

Define the functional $J : E \to R$ by

$$J(u) = \frac{1}{2}\int_0^L |u''(x)|^2 dx + \frac{1}{2}M\left(||u'||_2^2\right) - \int_0^L F(x, u(x))\, dx + G(u(L)), \tag{2.1}$$

where

$$M(t) = \textstyle\int_0^t m(s)\, ds,\ F(x,t) = \int_0^t f(x,s)\, ds,\ G(t) = \int_0^t g(s)\, ds.$$

By the continuity of the functions m, f and g, the functional J is of class C^1 and

$$\langle J'(u), \varphi \rangle = \int_0^L u''(x)\,\varphi''(x)\,dx + m\left(||u'||_2^2\right)\int_0^L u'(x)\,\varphi'(x)\,dx - \int_0^L f(x, u(x))\,\varphi(x)\,dx + g(u(L))\,\varphi(L),$$

for all u, $\varphi \in E$. Then we can deduce that $u \in E$ is a critical point of $J \Leftrightarrow$ it is a (classical) solution of the problem (1.1). This critical point is in fact a minimizer of J. The numerical results explain that u minimizes J as below.

test model	$J(u)$
$u - 0.01\sin x$	-11.031
u	-11.072
$u + 0.01\sin x$	-11.111

The study of the critical points of J can be done by using minimization arguments for weakly lower semicontinuous functionals in Ma [10]. A comprehensive introduction to variational problems for beam equations can be found in Grossinho and Tersian [3].

In this chapter, instead of solving the boundary value problem (1.1), we compute the minimizer of (2.1). There are many methods to do the job. First, J is discretized by the Finite Difference Method in section 2.2, and we use Mathematica build-in Solver and Newton's method to solve the nonlinear system in each subsections respectively. Then we apply the weighted residual Method to minimize J in section 1.3. In the end we will compare both methods to find the best one.

2.2 Finite Difference Method

Now we consider a typical computing model with

$$f(x) = -40x^3 + 48x^2 + 120x - 56,$$

$$m(s) = 1 + \frac{315}{353}s,$$

$$g(s) = 10s^3,$$

$$L = 1.$$

Its solution is

$$u(x) = x^5 - 2x^4 + 2x^2.$$

Let $0 = x_0 < x_1 < \cdots < x_n = L$ be a discretization of the interval $[0, L]$ with mesh size $h = x_i - x_{i-1}$. Let $u_i = u(x_i)$ and discetize

$$J(u) = \frac{1}{2}\int_0^L |u''(x)|^2 dx + \frac{1}{2}M\left(||u'||_2^2\right) - \int_0^L F(x, u(x))\, dx + G(u(L)),$$

as follows:

- $\int_0^L |u''(x)|^2 dx = \sum_{i=0}^{n-1} h\,[u''(x_i)]^2 = \sum_{i=0}^{n-1} h\left[\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}\right]^2$

$= \frac{1}{h^3}\sum_{i=0}^{n-1}(u_{i+1} - 2u_i + u_{i-1})^2.$

- $M\left(||u'||_2^2\right) = \int_0^{||u'||_2^2}\left(1 + \frac{315}{353}s\right)ds = s + \frac{315}{2\times 353}s^2|_0^{||u'||_2^2}$

$= \frac{1}{4h}\sum_{i=0}^{n-1}(u_{i+1} - u_{i-1})^2 + \frac{315}{11296h^2}\left(\sum_{i=0}^{n-1}(u_{i+1} - u_{i-1})^2\right)^2.$

- $||u'||_2^2 = \int_0^L (u')^2\, dx = \sum_{i=0}^{n-1} h\,[u'(x_i)]^2 = \frac{1}{4h}\sum_{i=0}^{n-1}[u_{i+1} - u_{i-1}]^2.$

- $F(x, u(x)) = \int_0^{u(x)} f ds = f \times s|_0^{u(x)} = f \times u,$

$\int_0^L F(x, u(x))\, dx = \int_0^L f(x)\, u(x)\, dx = \sum_{i=0}^{n-1} h f(x_i)\, u(x_i)$
$= h \sum_{i=0}^{n-1} u(x_i) \left[-40 (x_i)^3 + 48 (x_i)^2 + 120 x_i - 56 \right].$

- $G(u(L)) = G(u_n) = \int_0^{u_n} g(s)\, ds = \int_0^{u_n} 10 s^3 ds = \frac{10 s^4}{4} \Big|_0^{u_n} = \frac{5}{2} u_n^4.$

Boundary conditions become:

- $u(0) = u_0 = 0.$
- $u'(0) = 0 \Rightarrow \frac{u_1 - u_{-1}}{2h} = 0 \Rightarrow u_1 = u_{-1}.$

The minimizer of $\widehat{J}$ occurs where $\nabla \hat{J} = 0$, i.e. $\frac{\partial}{\partial u_i} \hat{J} = 0$, $i = 1, \ldots, n$, which is a system of nonlinear equations. In the following subsections, we use the mathematica build-in solver and Newton's method to solve this nonlinear system.

2.2.1 Mathematica Solver

In this subsection, we use Mathematica built-in function NSolve to slove the nonlinear system directly. The numerical result has multiple solutions, including real and complex. Only the ned soluion is required. The numerical errors of our computed solutions are given below.

n	Error
3	0.40263
4	0.28329
5	0.21958
6	0.17956
7	0.15200

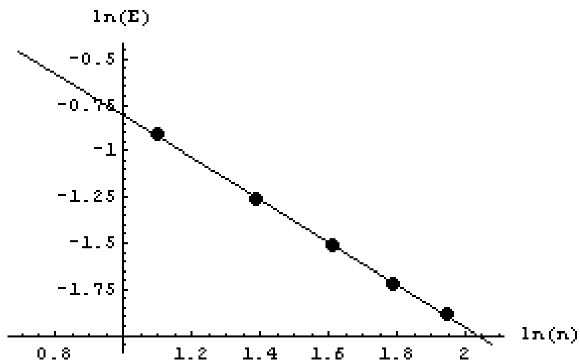

Figure 2.1

The asymptotic error formula is

$\ln E = 0.341414 - 1.14851 \ln(n)$

$\Rightarrow E = \frac{\exp^{0.341414}}{n^{1.14851}} = \frac{1.40694}{n^{1.14851}} = 1.40694 \times h^{1.14851} \approx O\left(h^{1.1}\right) \to 0$, as $h \to 0$.

Solving it is very expensive. This is becauses Mathematica produces many complex solutions, though only the unique real solution is needed. Despite of its convergence computing costs much time for large n. Consequently, we should use other effective methods to solve the problem.

2.2.2 Newton's Method

In this subsection, we use Newton's method to solve the nonlinear system discretized by the finite difference method: $F(u_1, u_2, \ldots, u_n) = \nabla \hat{J}$

- Newton's Algorithm:

Choose initial u^0. Repeat:
1. Compute Jacobian $A = \left[\frac{\partial F}{\partial u_i} \right]$ at $\left(u_1^{(k)}, \ldots, u_n^{(k)}\right)$.
2. Solve $A\mathbf{y}^{(k)} = F\left(u_1^{(k-1)}, u_2^{(k-1)}, \ldots, u_n^{(k-1)}\right)$ for $\mathbf{y}^{(k)}$.
3. $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} - \mathbf{y}^{(k)}$.

We use previous approximate solution as initial guess. Then the numerical results are given below.

n	Iteration numbers	Error
10	3	0.104224
20	3	0.051009
30	3	0.033784
40	3	0.025258
50	3	0.020169
60	3	0.016786
70	3	0.014376
80	3	0.012571

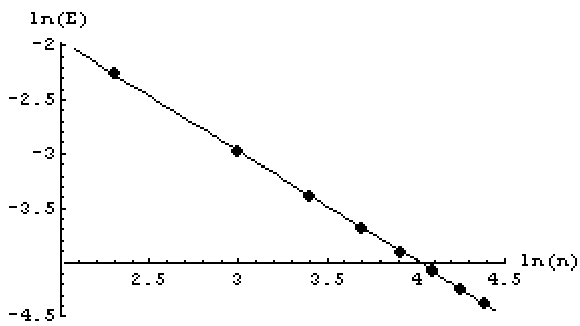

Figure 2.2

The asymptotic error formula is
$\ln E = 0.0725448 - 1.01617 \ln n$

$\Rightarrow E = \frac{\exp^{0.0725448}}{n^{1.01617}} = \frac{1.07524}{n^{1.01617}} = 1.07524 \times h^{1.01617} \approx O\left(h^{1.0}\right) \to 0$, as $h \to 0$.

We find that using Newton's method to solve the system is faster than using Mathematica built-in solver directly. Newton's method works even when $n = 80$, which costs only 30 minutes. But it is still time comsuming for large n, since each iteration needs a evaluation of Jacobi matrix and solving a $n \times n$ system.

2.3 Spectral Method

In this section, we use the spectral mathod to solve the minimization problem. Consider the critical point $\widehat{u}(x)$ of the form $\widehat{u}(x) = \sum_{k=0}^{n} c_k \phi_k(x)$. Substitute $\widehat{u}(x) = \sum_{k=0}^{n} c_k \phi_k(x)$ into $\hat{J}(u)$ to get

$\hat{J}(\widehat{u}) = \frac{1}{2}\int_0^L |\widehat{u}''(x)|^2 dx + \frac{1}{2}M(||\widehat{u}'||_2^2) - \int_0^L F(x, \widehat{u}(x))\,dx + G(\widehat{u}(L)) =$ $\frac{1}{2}\int_0^L \left(\sum_{k=0}^{n} c_k \phi_k''(x)\right)^2 dx + \frac{1}{2}M - \int_0^L F dx + G$, where $M = \int_0^{\left(\int_0^L \left(\sum_{k=0}^{n} c_k \phi_k'(x)\right)^2 dx\right)} m(s)\,ds$, $F = \int_0^{\left(\sum_{k=0}^{n} c_k \phi_k(x)\right)} f ds$, $G = \int_0^{\left(\sum_{k=0}^{n} c_k \phi_k(l)\right)} g(x)\,dx$.
The minimum of $\hat{J}(\widehat{u})$ occurs when $\nabla \hat{J}(\widehat{u}) = 0$, i.e.

$$\frac{\partial}{\partial c_k}\hat{J}(\widehat{u}) = 0,\ k = 1, \ldots, n. \tag{2.2}$$

By the constraints of boundary conditions, $u(0) = u'(0) = 0$, i.e.

$$\sum_{k=0}^{n} c_k \phi_k(0) = \sum_{k=0}^{n} c_k \phi_k'(0) = 0. \tag{2.3}$$

Combining (2.2)and (2.3), we derive a nonlinear system of $c_0, c_1, \ldots, c_n$. Typically consider the basis functions

$$\{\phi_k(x)\} = \left\{\sin\left(\tfrac{\pi}{2}kx\right), \cos\left(\tfrac{\pi}{2}kx\right)\right\}_{k=0}^{n},$$

so our numerical solution

$$\widehat{u}(x) = \sum_{k=1}^{n} c_k \sin\left(\tfrac{\pi}{2}kx\right) + \sum_{k=0}^{n} c_{n+1+k} \cos\left(\tfrac{\pi}{2}kx\right).$$

The numerical solutions contain real and complex ones, and we simply choose the real solution. We first use the built-in function NSolve in Mathematica to slove the nonlinear system directly. The errors are list below, where $n' = 2 \times n + 1$.

n'	Error
3	0.13371
5	0.00559
7	0.00023

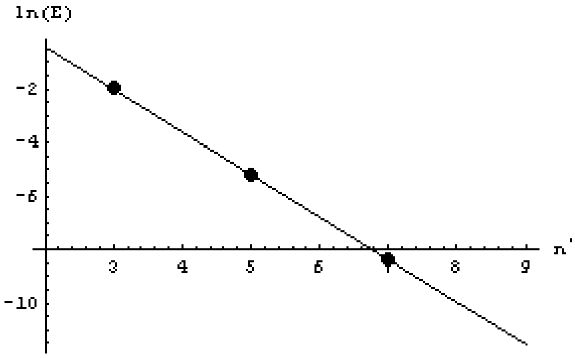

Figure 2.3

The asymptotic error formula is

$\ln E \cong 2.76459 - 1.59134^{n'}$

$\Rightarrow E = \left(e^{2.76459}\right)\left(e^{-1.59134}\right)^{n'} = 9.74212 \times 0.203653^{n'}$.

Next we use Newton's method to solve the nonlinear system. The numerical results are given below.

n'	Iteration numbers	Error
3	3	0.133717
5	10	0.005592
7	9	0.000233
9	10	9.6×10^{-6}
11	13	3.5×10^{-7}
13	15	1.4×10^{-8}
15	31	5.8×10^{-10}

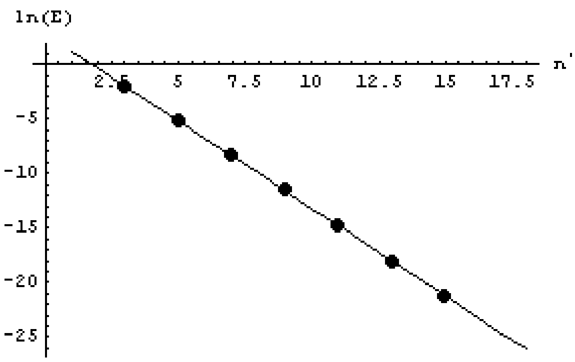

Figure 2.4

The asymptotic error formula is

$\ln E \cong 2.85545 - 1.60829 n'$

$\Rightarrow E = \left(e^{2.85545}\right)\left(e^{-1.60829}\right)^{n'} = 17.3823 \times 0.20023^{n'}$, where $n' = 2 \times n + 1$.

We remark that the basis $\{\phi_k(x)\}$ must be chosen properly to make the method converge. For example, $\{\phi_k(x)\} = \{\sin(2k\pi x), \cos(2k\pi x)\}$ is a bad basis. The reason is that $\widehat{u}(1) = \sum_{i=0}^{n} c_i \phi_i(1) = 0$, which is impossible to approximate $u(1) = x^5 - 2x^4 + 2x^2|_{x=1} = 1$.

Due to the exponential convergence of the spectral method, it is faster than the previous finite difference method. For solving the discretized nonlinear system, Newton's method is the best choice.

Chapter 3

Methods for Integral Equation

3.1 Introduction

An integral equation is an equation in which the unknown function appears under an integral sign. Linear integral equations of most frequent occurrence in practice are conventionally divided into the Fredholm equation

$$\alpha(x)\, y(x) = F(x) + \lambda \int_a^b K(x,\xi)\, y(\xi)\, d\xi,$$

and the Volterra equation

$$\alpha(x)\, y(x) = F(x) + \lambda \int_a^x K(x,\xi)\, y(\xi)\, d\xi.$$

See Hildebrand [8] for more detail.

A differential equation can be converted to a integral equation naturally. As a typical example, the first order initial value problem

$$\begin{cases} y'(x) = f(x, y(x)), \\ y(a) = \alpha, \end{cases}$$

is equivalent to the integral equation

$$y(x) = \alpha + \int_a^x f(x, y(t))\, dt.$$

Similarly for the second order initial value problem

$$\begin{cases} \frac{d^2y}{dx} + \lambda y = f(x), \\ y(0) = 1, \ y'(0) = 0, \end{cases}$$

it corresponds to the integral equation

$$y(x) = \lambda \int_a^x (\xi - x)\, y(\xi)\, d\xi + 1 - \int_a^x (\xi - x)\, f(\xi)\, d\xi.$$

In this chapter, We again study the nonlinear 4-th order beam equations, coupled with nonlinear boundary conditions. The well-posedness of a simple boundary value problem, depending on various type of boundary conditions, will be discussed. Then we transform the original problem to the corresponding integral equation and design different numerical schemes to solve this kind of integral equation. These methods can be found in the book of Hildebrand [8]. We will compare all methods to find the best one in the end.

This chapter is organized as follows. The existence and uniqueness theorems are presented in section 3.2. We solve the corresponding integral equation by Picard's Iteration in section 3.3, Finite Difference Method in section 3.4, Collocation Method in section 3.5, and Weighted Residual Method in section 3.6. We state the conclusion in the final section 3.7.

3.2 Existence and Uniqueness

Consider the simplest 4-th order differential equation

$$u^{(4)}(x) = f(x, u(x)) \text{ for } x \in [0, L]\,. \tag{3.1}$$

Its solution has the form

$$u(x) = R\int_0^x (x-t)^3 f(t, u\,(t))dt + Ax^3 + Bx^2 + Cx + D.$$

This is an integral equation of $u(x)$. The constant R can be found explicitly
$u(x) = R\int_0^x (x-t)^3 f(t, u\,(t))dt + Ax^3 + Bx^2 + Cx + D$
$\Rightarrow u^{(4)}(x) = 6Rf(x, u\,(x))$
$\Rightarrow R = \frac{1}{6}$.
The constants A, B, C, D are determined by the boundary conditions. The detail is described below.

The existence of solutions strongly depend on the boundary conditions. We will divide the solutions into well-posed and ill-posed. We first state the well posed cases as the following 11 types.

Type 1:

Notice that

$$\begin{cases} u(0) = \alpha,\ u'(0) = \beta \\ u(1) = \gamma,\ u'(1) = \delta \end{cases}$$

$$u(0) = \alpha \Rightarrow D = \alpha.$$

$$u'(0) = \beta \Rightarrow C = \beta.$$

$$u(1) = \frac{1}{6}\int_0^1 (1-t)^3 f(t, u\,(t))dt + A + B + \beta + \alpha = \gamma,$$

$$u'(1) = \frac{1}{6}\int_0^1 3(1-t)^2 f(t, u\,(t))dt + 3A + 2B + \beta = \delta. \tag{3.2}$$

Substract (3.2) from

$$3u(1) = \frac{1}{6}\int_0^1 3(1-t)^3 f(t, u\,(t))dt + 3A + 3B + 3\beta + 3\alpha = 3\gamma$$

to get

$$B = \frac{1}{2}\int_0^1 t(1-t)^2 f(t, u\,(t))dt + 3\gamma - \delta - 3\alpha - 2\beta.$$

Similarly, substract (3.2) from

$$2u(1) = \frac{1}{6}\int_0^1 2(1-t)^3 f(t, u\,(t))dt + 2A + 2B + 2\beta + 2\alpha = 2\gamma$$

to get

$$A = -\int_0^1 \left(\frac{1}{6} + \frac{1}{3}t\right)(1-t)^2 f(t,u(t))dt + \delta + 2\alpha + \beta - 2\gamma.$$

Type 2:
Notice that

$$\begin{cases} u(0) = \alpha, \ u'(0) = \beta \\ u(1) = \gamma, \ u''(1) = \delta \end{cases}$$

$$u(0) = \alpha \Rightarrow D = \alpha.$$

$$u'(0) = \beta \Rightarrow C = \beta.$$

$$u(1) = \frac{1}{6}\int_0^1 (1-t)^3 f(t,u(t))dt + A + B + \beta + \alpha = \gamma.$$

$$u''(1) = \frac{1}{6}\int_0^1 6(1-t)f(t,u(t))dt + 6A + 2B = \delta \tag{3.3}$$

Substract (3.3) from

$$6u(1) = \int_0^1 (1-t)^3 f(t,u(t))dt + 6A + 6B + 6\beta + 6\alpha = 6\gamma$$

to get

$$B = -\frac{1}{4}\int_0^1 (t^2 - 2t)(1-t) f(t,u(t))dt + 6\gamma - 6\delta - 6\alpha - \beta.$$

Similarly, substract (3.3) from

$$2u(1) = \frac{1}{3}\int_0^1 (1-t)^3 f(t,u(t))dt + 2A + 2B + 2\beta + 2\alpha = 2\gamma$$

to get

$$A = -\frac{1}{4}\int_0^1 (1-t)\left(\frac{2}{3} + \frac{2}{3}t - \frac{1}{3}t^2\right) f(t,u(t))dt + \delta + 2\alpha + 2\beta - 2\gamma.$$

Type 3:
Notice that

$$\begin{cases} u(0) = \alpha, \ u'(0) = \beta \\ u(1) = \gamma, \ u'''(1) = \delta \end{cases}$$

$$u(0) = \alpha \Rightarrow D = \alpha.$$

$$u'(0) = \beta \Rightarrow C = \beta.$$

$$u(1) = \frac{1}{6}\int_0^1 (1-t)^3 f(t,u(t))dt + A + B + \beta + \alpha = \gamma.$$

$$u'''(1) = \int_0^1 f(t,u(t))dt + 6A = \delta$$

$$B = \frac{1}{6}\left(\int_0^1 t(3 - 3t + t^2) f(t,u(t))dt - \delta\right) + \gamma - \delta - \beta.$$

$$A = \frac{1}{6}\left(\delta - \int_0^1 f(t,u(t))dt\right).$$

Type 4:
Notice that

$$\begin{cases} u(0) = \alpha,\ u'(0) = \beta \\ u'(1) = \gamma,\ u''(1) = \delta \end{cases}$$
$$u(0) = \alpha \Rightarrow D = \alpha.$$
$$u'(0) = \beta \Rightarrow C = \beta.$$

$$u'(1) = \frac{1}{2}\int_0^1 (1-t)^2 f(t, u(t))dt + 3A + 2B + \beta = \gamma \tag{3.4}$$

Substract (3.4) from

$$u''(1) = \int_0^1 (1-t) f(t, u(t))dt + 6A + 2B = \delta$$

to get

$$A = -\tfrac{1}{3}\int_0^1 (1-t)\left(\tfrac{1}{2} + \tfrac{1}{2}t\right) f(t, u(t))dt + \delta + \beta - \gamma$$

$$2u'(1) = \int_0^1 (1-t)^2 f(t, u(t))dt + 6A + 4B + 2\beta = 2\gamma \tag{3.5}$$

Similarly, substract (3.5) from

$$u''(1) = \int_0^1 (1-t) f(t, u(t))dt + 6A + 2B = \delta$$

to get

$$B = \tfrac{1}{2}\left(\int_0^1 t(1-t) f(t, u(t))dt + 2\gamma - \delta - 2\beta\right).$$

Type 5:
Notice that

$$\begin{cases} u(0) = \alpha,\ u'(0) = \beta \\ u'(1) = \gamma,\ u'''(1) = \delta \end{cases}$$
$$u(0) = \alpha \Rightarrow D = \alpha.$$
$$u'(0) = \beta \Rightarrow C = \beta.$$
$$u'(1) = \tfrac{1}{2}\int_0^1 (1-t)^2 f(t, u(t))dt + 3A + 2B + \beta = r$$
$$u'''(1) = \int_0^1 f(t, u(t))dt + 6A = \delta.$$
$$\Rightarrow A = \tfrac{1}{6}\left(\delta - \int_0^1 f(t, u(t))dt\right).$$
$$B = \tfrac{1}{4}\left(\int_0^1 t(2-t) f(t, u(t))dt\right) + \tfrac{1}{4}\left(2\gamma - \delta - 2\beta\right).$$

Type 6:
Notice that

$$\begin{cases} u(0)=\alpha,\ u'(0)=\beta \\ u''(1)=\gamma,\ u'''(1)=\delta \end{cases}$$

$$u(0)=\alpha \Rightarrow D=\alpha.$$
$$u'(0)=\beta \Rightarrow C=\beta.$$
$$u''(1)=\int_0^1 (1-t)f(t,u\,(t))dt+6A+2B=\gamma.$$
$$u'''(1)=\int_0^1 f(t,u\,(t))dt+6A=\delta$$
$$\Rightarrow A=\frac{1}{6}\left(\delta-\int_0^1 f(t,u\,(t))dt\right).$$
$$u''(1)=\int_0^1 (-t)f(t,u\,(t))dt+\delta+2B=\gamma$$
$$B=\frac{1}{2}\left(\int_0^1 tf(t,u\,(t))dt+\gamma-\delta\right).$$

Type 7:
Notice that

$$\begin{cases} u(0)=\alpha,\ u''(0)=\beta \\ u(1)=\gamma,\ u''(1)=\delta \end{cases}$$

$$u(0)=\alpha \Rightarrow D=\alpha.$$
$$u''(0)=2B=\beta \Rightarrow B=\frac{1}{2}\beta.$$
$$u(1)=\frac{1}{6}\int_0^1 (1-t)^3 f(t,u\,(t))dt+A+\frac{1}{2}\beta+C+\alpha=\gamma.$$
$$u''(1)=\int_0^1 (1-t)f(t,u\,(t))dt+6A+\beta=\delta$$
$$\Rightarrow A=\frac{\int_0^1 (t-1)f(t,u(t))dt+\delta-\beta}{6}.$$
$$C=r-\frac{1}{6}\int_0^1 (1-t)\left(-2t+t^2\right)f(t,u\,(t))dt-\frac{\delta}{6}-\frac{\beta}{3}-\alpha.$$

Type 8:
Notice that

$$\begin{cases} u(0)=\alpha,\ u''(0)=\beta \\ u(1)=\gamma,\ u'''(1)=\delta \end{cases}$$

$$u(0)=\alpha \Rightarrow D=\alpha.$$
$$u''(0)=\beta \Rightarrow B=\frac{\beta}{2}.$$
$$u(1)=\frac{1}{6}\int_0^1 (1-t)^3 f(t,u\,(t))dt+A+\frac{\beta}{2}+c+\alpha=\gamma.$$
$$u'''(1)=\int_0^1 f(t,u\,(t))dt+6A=\delta$$
$$\Rightarrow A=\frac{\delta-\int_0^1 f(t,u(t))dt}{6}.$$
$$C=r-\frac{1}{6}\int_0^1 (1-t)^3 f(t,u\,(t))dt+\frac{1}{6}\int_0^1 f(t,u\,(t))dt-\frac{\delta}{6}-\frac{\beta}{2}-\alpha=$$
$$r+\frac{1}{6}\int_0^1 \left(1-(1-t)^3\right)f(t,u\,(t))dt-\frac{\delta}{6}-\frac{\beta}{2}-\alpha.$$

Type 9:
Notice that

$$\begin{cases} u(0)=\alpha,\ u''(0)=\beta \\ u'(1)=\gamma,\ u''(1)=\delta \end{cases}$$

$$u(0)=\alpha \Rightarrow D=\alpha.$$

$$u''(0)=\beta \Rightarrow B=\tfrac{\beta}{2}.$$

$$u'(1)=\tfrac{1}{2}\int_0^1(1-t)^2 f(t,u(t))dt+3A+B+C=\gamma$$

$$u''(1)=\int_0^1(1-t)f(t,u(t))dt+6A+\beta=\delta$$

$$\Rightarrow A=\frac{-\int_0^1(1-t)f(t,u(t))dt+\delta-\beta}{6}.$$

$$C=\tfrac{1}{2}\int_0^1 t(1-t)f(t,u(t))dt-\tfrac{\delta}{2}-\tfrac{\beta}{2}+\gamma.$$

Type 10:
Notice that

$$\begin{cases} u(0)=\alpha,\ u''(0)=\beta \\ u'(1)=\gamma,\ u'''(1)=\delta \end{cases}$$

$$u(0)=\alpha \Rightarrow D=\alpha.$$

$$u''(0)=\beta \Rightarrow B=\tfrac{\beta}{2}.$$

$$u'(1)=\tfrac{1}{2}\int_0^1(1-t)^2 f(t,u(t))dt+3A+\beta+C=\gamma.$$

$$u'''(1)=\int_0^1 f(t,u(t))dt+6A=\delta$$

$$\Rightarrow A=\frac{\delta-\int_0^1 f(t,u(t))dt}{6}.$$

$$C=\tfrac{1}{2}\int_0^1(2t-t^2)f(t,u(t))dt-\tfrac{\delta}{2}-\beta+\gamma.$$

Type 11:
Notice that

$$\begin{cases} u(0)=\alpha,\ u'''(0)=\beta \\ u'(1)=\gamma,\ u''(1)=\delta \end{cases}$$

$$u(0)=\alpha \Rightarrow D=\alpha.$$

$$u'''(0)=\beta \Rightarrow A=\tfrac{\beta}{6}.$$

$$u'(1)=\tfrac{1}{2}\int_0^1(1-t)^2 f(t,u(t))dt+2B+\tfrac{\beta}{2}+C=\gamma$$

$$u''(1)=\int_0^1(1-t)f(t,u(t))dt+2B+\beta=\delta$$

$$B=\tfrac{1}{2}\left(\delta-\beta-\int_0^1(1-t)f(t,u(t))dt\right).$$

$$C=\int_0^1(1-t)\left(\tfrac{1}{2}t+\tfrac{1}{2}\right)f(t,u(t))dt-\delta+\tfrac{\beta}{2}+\gamma.$$

In the sequel, we consider the ill-posed boundary value problems. The existence and non-uniquences theorem of each type of boundary conditions are stated as following.

Theorem 1 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x, u(x)), \\ u''(0) = \alpha, \ u'''(0) = \beta, \\ u''(1) = \gamma, \ u'''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 (1-t) f(t, u(t)) dt + \beta + \alpha = \gamma, \quad \int_0^1 f(t, u(t)) dt + \beta = \delta.$$

(ii)In the affirmative case, its general solution has the form

$$u(x) = \frac{1}{6} \int_0^x (x-t)^3 f(t, u(t)) dt + \frac{\beta}{6} x^3 + \frac{\alpha}{2} x^2 + cx + d,$$

where c and d are constants.

(iii)Moreover, if $u(0)$ and $u'(0)$ are prescribed, then the solution exists and is unique. In fact

$$u(x) = \frac{1}{6} \int_0^x (x-t)^3 f(t, u(t)) dt + \frac{\beta}{6} x^3 + \frac{\alpha}{2} x^2 + u'(0)x + u(0).$$

Theorem 2 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x, u(x)), \\ u'(0) = \alpha, \ u'''(0) = \beta, \\ u'(1) = \gamma, \ u'''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 f(t, u(t)) dt + \beta = \delta.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6} \int_0^x (x-t)^3 f(t, u(t)) dt + \frac{\beta}{6} x^3 \\ &\quad + \left(\gamma - \frac{1}{2} \int_0^1 (1-t)^2 f(t, u(t)) dt - \frac{\beta}{2} - \alpha\right) x^2 + \alpha x + d, \end{aligned}$$

where d are constants.

(iii)Moreover, if $u(0)$ is prescribed, then the solution exists and is unique. In fact

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \frac{\beta}{6}x^3 \\ &+ \left(\gamma - \frac{1}{2}\int_0^1 (1-t)^2 f(t,u(t))dt - \frac{\beta}{2} - \alpha\right)x^2 + \alpha x + u(0). \end{aligned}$$

Theorem 3 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x,u(x)), \\ u'(0) = \alpha,\ u''(0) = \beta, \\ u'(1) = \gamma,\ u''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 (1-t)f(t,u(t))dt + 6\left(\gamma - \frac{1}{2}\int_0^1 (1-t)^2 f(t,u(t))dt - \beta - \alpha\right) + \beta = \delta.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \left(\gamma - \frac{1}{2}\int_0^1 (1-t)^2 f(t,u(t))dt - \beta - \alpha\right)x^3 \\ &+ \frac{\beta}{2}x^2 + \alpha x + d. \end{aligned}$$

where d are constants.

(iii)Moreover, if $u(0)$ is prescribed, then the solution exists and is unique. In fact

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \left(\gamma - \frac{1}{2}\int_0^1 (1-t)^2 f(t,u(t))dt - \beta - \alpha\right)x^3 \\ &+ \frac{\beta}{2}x^2 + \alpha x + u(0). \end{aligned}$$

Theorem 4 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x,u(x)), \\ u(0) = \alpha,\ u'''(0) = \beta, \\ u(1) = \gamma,\ u'''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 f(t,u(t))dt+\beta=\delta.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt+\frac{\beta}{6}x^3+bx^2 \\ &\quad +\left(\gamma-\int_0^1(1-t)^3 f(t,u(t))dt-\frac{\beta}{6}-b-\alpha\right)x+\alpha. \end{aligned}$$

where b are constants.

(iii)Moreover, if $u''(0)$ is prescribed, then the solution exists and is unique. In fact
$u(x)=\frac{1}{6}\int_0^x(x-t)^3 f(t,u(t))dt+\frac{\beta}{6}x^3+\frac{u''(0)}{2}x^2+\left(\gamma-\int_0^1(1-t)^3 f(t,u(t))dt-\frac{\beta}{6}-\frac{u''(0)}{2}-\alpha\right)x$
$+\alpha$.

Theorem 5 *Consider*

$$\begin{cases} u^{(4)}(x)=f(x,u(x)), \\ u(0)=\alpha,\ u''(0)=\beta, \\ u''(1)=\gamma,\ u'''(1)=\delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 (1-t)\,f(t,u(t))dt+\delta-\int_0^1 f(t,u(t))dt+\beta=\gamma.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt+\frac{1}{6}\left(\delta-\int_0^1 f(t,u(t))dt\right)x^3 \\ &\quad +\frac{\beta}{2}x^2+cx+\alpha. \end{aligned}$$

where c are constants.

(iii)Moreover, if $u'(0)$ is prescribed, then the solution exists and is unique. In fact

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt+\frac{1}{6}\left(\delta-\int_0^1 f(t,u(t))dt\right)x^3 \\ &\quad +\frac{\beta}{2}x^2+u'(0)x+\alpha. \end{aligned}$$

Theorem 6 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x, u(x)), \\ u(0) = \alpha, \ u'''(0) = \beta, \\ u''(1) = \gamma, \ u'''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 f(t, u(t))dt + \beta = \delta.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t, u(t))dt + \frac{\beta}{6}x^3 \\ &+\frac{1}{2}\left(\gamma - \int_0^1 (1-t) f(t, u(t))dt - \beta\right) x^2 + cx + \alpha \end{aligned}$$

where c are constants.

(iii)Moreover, if $u'(0)$ is prescribed, then the solution exists and is unique. In fact

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t, u(t))dt + \frac{\beta}{6}x^3 \\ &+\frac{1}{2}\left(\gamma - \int_0^1 (1-t) f(t, u(t))dt - \beta\right) x^2 + u'(0)x + \alpha. \end{aligned}$$

Theorem 7 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x, u(x)), \\ u'(0) = \alpha, \ u''(0) = \beta, \\ u''(1) = \gamma, \ u'''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 f(t, u(t))dt + \gamma - \int_0^1 (1-t) f(t, u(t))dt - \beta = \delta.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t, u(t))dt + \frac{1}{6}\left(\gamma - \int_0^1 (1-t) f(t, u(t))dt - \beta\right) x^3 \\ &+\frac{\beta}{2}x^2 + \alpha x + d. \end{aligned}$$

where d are constants.

(iii)Moreover, if $u(0)$ is prescribed, then the solution exists and is unique. In fact

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \frac{1}{6}\left(\gamma - \int_0^1 (1-t)f(t,u(t))dt - \beta\right)x^3 \\ &+\frac{\beta}{2}x^2 + \alpha x + u(0). \end{aligned}$$

Theorem 8 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x,u(x)), \\ u'(0) = \alpha,\ u'''(0) = \beta, \\ u''(1) = \gamma,\ u'''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 f(t,u(t))dt + \beta = \delta.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \frac{\beta}{6}x^3 + \frac{1}{2}\left(\gamma - \int_0^1 (1-t)f(t,u(t))dt - \beta\right)x^2 \\ &+\alpha x + d. \end{aligned}$$

where d are constants.

(iii)Moreover, if $u(0)$ is prescribed, then the solution exists and is unique. In fact

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \frac{\beta}{6}x^3 + \frac{1}{2}\left(\gamma - \int_0^1 (1-t)f(t,u(t))dt - \beta\right)x^2 \\ &+\alpha x + u(0). \end{aligned}$$

Theorem 9 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x,u(x)), \\ u(0) = \alpha,\ u'''(0) = \beta, \\ u'(1) = \gamma,\ u'''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 f(t,u(t))dt + \beta = \delta.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \frac{\beta}{6}x^3 + bx^2 \\ &+ \left(\frac{\gamma}{2} - \frac{1}{4}\int_0^1 (1-t)^2 f(t,u(t))dt - \frac{\beta}{4} - b\right)x + \alpha. \end{aligned}$$

where b are constants.

(iii)Moreover, if $u''(0)$ is prescribed, then the solution exists and is unique. In fact

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \frac{\beta}{6}x^3 + \frac{u''(0)}{2}x^2 \\ &+ \left(\frac{\gamma}{2} - \frac{1}{4}\int_0^1 (1-t)^2 f(t,u(t))dt - \frac{\beta}{4} - \frac{u''(0)}{2}\right)x + \alpha. \end{aligned}$$

Theorem 10 *Consider*

$$\begin{cases} u^{(4)}(x) = f(x,u(x)), \\ u'(0) = \alpha, \ u''(0) = \beta, \\ u'(1) = \gamma, \ u'''(1) = \delta. \end{cases}$$

(i) This problem has a solution if and only if

$$\int_0^1 f(t,u(t))dt + 2\left(\gamma - \left(\frac{1}{2}\int_0^1 (1-t)^2 f(t,u(t))dt + \beta + \alpha\right)\right) = \delta.$$

(ii)In the affirmative case, its general solution has the form

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \frac{1}{3}\left(\gamma - \frac{1}{2}\int_0^1 (1-t)^2 f(t,u(t))dt - \beta - \alpha\right)x^3 \\ &+ \frac{\beta}{2}x^2 + \alpha x + d. \end{aligned}$$

where d are constants.

(iii)Moreover, if $u(0)$ is prescribed, then the solution exists and is unique. In fact

$$\begin{aligned} u(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + \frac{1}{3}\left(\gamma - \frac{1}{2}\int_0^1 (1-t)^2 f(t,u(t))dt - \beta - \alpha\right)x^3 \\ &+ \frac{\beta}{2}x^2 + \alpha x + u(0). \end{aligned}$$

3.3 Picard's Iteration

Picard's iteration is also knowns as the method of successive approximation or the fixed point iteration. It is a very simple method to compute the solution of an integral equation, and we will use it in this subsection.

We introduce the method for a simple example,

$$y'(x) = y(y(x)), \, y(\alpha) = \alpha.$$

First, we transform the differential equation to an integral equation

$$\begin{gathered} \int_\alpha^x y'(s)\,ds = \int_\alpha^x y(y(s))\,ds \\ \Rightarrow y(x) - y(\alpha) = \int_\alpha^x y(y(s))\,ds \\ \Rightarrow y(x) = \alpha + \int_\alpha^x y(y(s))\,ds. \end{gathered}$$

Picard's method chooses initial $y_0(t)$ and repeat

$$y_{n+1}(x) = \alpha + \int_\alpha^x y_n(y_n(s))\,ds.$$

For example, if $y_0 = \alpha$, then

$$\begin{gathered} y_1(x) = \alpha + \int_\alpha^x \alpha ds = \alpha + \alpha(x-\alpha), \\ y_2(x) = \alpha + \int_\alpha^x \alpha + \alpha^2(s-\alpha)\,ds = \alpha + \alpha(x-\alpha) + \tfrac{\alpha^2}{2}(x-\alpha)^2, \\ \vdots \end{gathered}$$

Now we consider the first nonlinear problem

$$\begin{cases} u^{(4)}(x) = f(x, u(x)), \\ u''(0) = u''(1) = 0, \\ u'''(0) = g(u(0)), \;\; u'''(1) = h(u(1)). \end{cases}$$

Suppose the compatibility condition

$$h(u(1)) - g(u(0)) = \int_0^1 f(t, u(t))dt$$

holds. Then, the solution satisfies

$$u(x) = \frac{1}{6}\int_0^1 K(x,t)f(t,u(t))dt + u(0)(1-x) + u(1)x, \tag{3.6}$$

where

$$K(x,t)=\left\{\begin{array}{l} x^3t+xt^3+2xt-3x^2t-t^3,\ 0\le t\le x\le 1 \\ x^3t+xt^3+2xt-3xt^2-x^3,\ 0\le x\le t\le 1 \end{array}\right. ,$$

$$u(0)=g^{-1}\left(\int_0^1 (t-1)f(t,u(t))dt\right),\ u(1)=h^{-1}\left(\int_0^1 tf(t,u(t))dt\right).$$

A model problem is

$$\begin{aligned} f(x,w) &= \frac{1}{14400}x^{10}-\frac{1}{720}x^9+\frac{1}{108}x^8-\frac{5}{216}x^7+\frac{1361}{64800}x^6 \\ &\quad -\frac{67}{2160}x^5+\frac{217}{1296}x^4-\frac{25}{108}x^3+\frac{1369}{129600}x^2 \\ &\quad -\frac{253}{216}x+\frac{97}{36}-w^2. \end{aligned}$$

The bearings are represented by

$$g(w)=-w,\ h(w)=\frac{27}{12}w^3.$$

The exact solution is

$$u(x)=-\tfrac{1}{120}x^5+\tfrac{1}{12}x^4-\tfrac{5}{36}x^3-\tfrac{37}{360}x+\tfrac{5}{6},$$

with

$$\begin{array}{l} u^{(4)}(x)=2-x, \\ u(0)=5/6, \\ u(1)=2/3, \\ u''(0)=u''(1)=0, \\ u'''(0)=-5/6, \\ u'''(1)=2/3. \end{array}$$

Its solution $u(x)$ satisfies the integral equation

$$u(x)=\tfrac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt+Ax^3+Bx^2+Cx+D.$$

The constants A,B,C,D are computed by the boundary conditions.

- $u''(0)=0 \Rightarrow B=0$
- $u''(1)=0$

$\Rightarrow \frac{1}{6}\int_0^1 6(1-t)f(t,u\,(t))dt + 6A + 2B = 0$
$\Rightarrow A = \frac{1}{6}\int_0^1 (t-1)f(t,u\,(t))dt.$

- $u'''(0) = g(u(0))$

$\Rightarrow u'''(0) = -u(0) = -D$
$\Rightarrow D = -6A$

- $u'''(1) = h(u(1)) \Rightarrow u'''(1) = \frac{27}{12}\left[u(1)\right]^3$

$\Rightarrow \frac{1}{6}\int_0^1 6f(t,u\,(t))dt + 6A = \frac{27}{12}\left[u(1)\right]^3$
$\Rightarrow \int_0^1 f(t,u\,(t))dt + \int_0^1 (t-1)f(t,u\,(t))dt = \frac{27}{12}\left[u(1)\right]^3$
$\Rightarrow \int_0^1 tf(t,u\,(t))dt = \frac{27}{12}\left[u(1)\right]^3$
$\Rightarrow u(1) = \left[\frac{12}{27}\int_0^1 tf(t,u\,(t))dt\right]^{\frac{1}{3}}$
$\Rightarrow \frac{1}{6}\int_0^1 (1-t)^3 f(t,u\,(t))dt + A + C + D = \left[\frac{12}{27}\int_0^1 tf(t,u\,(t))dt\right]^{\frac{1}{3}}$
$\Rightarrow C = 5A + \left[\frac{12}{27}\int_0^1 tf(t,u\,(t))dt\right]^{\frac{1}{3}} - \frac{1}{6}\int_0^1 (1-t)^3 f(t,u\,(t))dt.$
Therefore We get

$$\begin{aligned} u\,(x) &= \{\frac{1}{6}\int_0^x (x-t)^3 f(t,u\,(t))dt + Ax^3 \\ &\quad +\{5A + \left[\frac{12}{27}\int_0^1 tf(t,u\,(t))dt\right]^{\frac{1}{3}} \\ &\quad -\frac{1}{6}\int_0^1 (1-t)^3 f(t,u\,(t))dt\}x - 6A\}, \\ \text{where } A &= \frac{1}{6}\left[\int_0^1 (t-1)f(t,u\,(t))dt\right]. \end{aligned}$$

The Picard's method for this problem is stated below.
Picard's Iteration:
1.Choose the initial $u_0\,(x)$.
2.Repeat

$$\begin{aligned} u_k\,(x) &= \frac{1}{6}\int_0^x (x-t)^3 f(t,u_{k-1}\,(t))dt + A_k x^3 \\ &\quad +\{5A_k + \left[\frac{12}{27}\int_0^1 tf(t,u_{k-1}\,(t))dt\right]^{\frac{1}{3}} \end{aligned}$$

$$-\frac{1}{6}\int_0^1 (1-t)^3 f(t, u_{k-1}(t))dt\}x - 6A_k \left(\text{mod } x^{k+1}\right),$$

$$\text{where } A_k = \frac{1}{6}\left[\int_0^1 (t-1) f(t, u_{k-1}(t))dt\right].$$

If constant initial u_0 is choosen such that $|u_0(x)| > 1.3$ then Picard's iteration fails to converge. We use $u_0(x) = 0$ as initial guess here. Then the numerical results are given below.

k	Error
10	3.1×10^{-4}
20	1.8×10^{-6}
30	1.0×10^{-8}
40	6.1×10^{-11}
50	3.5×10^{-13}
60	2.2×10^{-15}

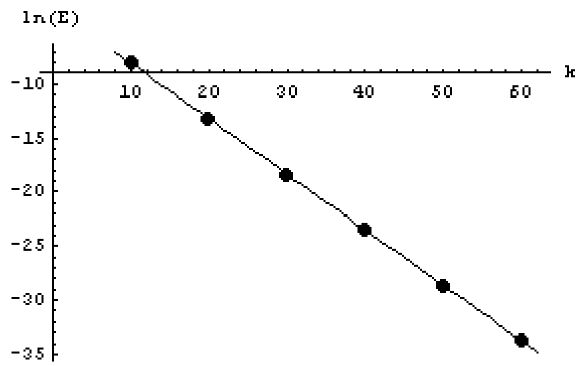

Figure 3.1

The asymptotic error formula is

$\ln E = -2.96487 - 0.513759k$

$\Rightarrow E = \left(e^{-2.96487}\right)\left(e^{-0.513759}\right)^k = 0.0515672 \times 0.598243^k.$

Next, we consider the second beam problem in [11]

$$u^{(4)}(x) - M\left(\int_0^L |u'(x)|^2 dx\right) u''(x) = f(x, u(x)),\ 0 < x < L,$$

$$u(0) = u''(0) = u(L) = 0,$$

$$u''(L) = g(u'(L)),$$

where $M \in C(\mathbf{R}^+)$, $f \in C([0, L] \times \mathbf{R})$ and $g \in C(\mathbf{R})$ are real functions.

Let us reduce the problem to a coupled second order system in the following way.

$$\begin{cases} u'' = M\left(||u'||_2^2\right) u + v, \\ u(0) = u(L) = 0, \end{cases}$$

and

$$\begin{cases} v'' = f(x, u), \\ v(0) = 0, \ v(L) = g(u'(L)). \end{cases}$$

Let G be the Green's function defined by

$$G(x,t) = \begin{cases} \frac{t(x-L)}{L}, & 0 \leq t \leq x \leq L, \\ \frac{x(t-L)}{L}, & 0 \leq x \leq t \leq L. \end{cases}$$

Then for each continuous h, $w(x) = \int_0^L G(x,t)\, h(t)\, dt$ is a solution of the Dirichlet problem $w'' = h$ with $w(0) = w(L) = 0$. We conclude the

$$u(x) = \int_0^L G(x,t) \left[M\left(||u'||_2^2\right) u(t) + v(t)\right] dt,$$

where

$$v(t) = \int_0^L G(t,s)\, f(s, u(s))\, ds + \frac{t}{L} g(u'(L)).$$

We have added the term $\frac{t}{L} g(u'(L))$ because $v(t) = \int_0^L G(t,s)\, f(s, u(s))\, ds$ give only $v(0) = v(L) = 0$. Hence we see that **u** is a solution of the problem if and only if it is a fixed point of

$$\begin{aligned} Tu(x) \quad = \quad & \int_0^L \int_0^L G(x,t)\, G(t,s)\, f(s, u(s))\, ds + \frac{g(u'(L))}{L} \int_0^L G(x,t)\, t dt \\ & + M\left(||u'||_2^2\right) \int_0^L G(x,t)\, u(t)\, dt. \end{aligned}$$

Our model problem selects

$$f(x,u) = x^5 - x^4 - 21\,x^3 + 12\,x^2 + 127\,x - 24 - u.$$

The bearings are represented by

$$g(s) = -2s, \; M(s) = \frac{1}{2} + \frac{2025}{1352}s^2.$$

The exact solution in $[0,1]$ is

$$u(x) = x^5 - x^4 - x^3 + x.$$

Some computational details are listed below:

- $\int_0^1 G(t,s)\, f(s,u(s))\, ds = \int_0^t G(t,s)\, f(s,u(s))\, ds + \int_t^1 G(t,s)\, f(s,u(s))\, ds.$
- $\int_0^1 \int_0^1 G(x,t) \left[G(t,s)\, f(s,u(s))\, ds\right] dt$

$= \int_0^x G(x,t) \left[\int_0^t G(t,s)\, f(s,u(s))\, ds + \int_t^1 G(t,s)\, f(s,u(s))\, ds\right] dt$
$+ \int_x^1 G(x,t) \left[\int_0^t G(t,s)\, f(s,u(s))\, ds + \int_t^1 G(t,s)\, f(s,u(s))\, ds\right] dt.$

- $g(u'(1)) \left(\int_0^1 G(x,t)\, t dt\right) = -2(u'(1)) \left[\int_0^x G(x,t)\, t dt + \int_x^1 G(x,t)\, t dt\right].$
- $M(||u'||_2^2) \left(\int_0^1 G(x,t)\, u(t)\, dt\right) = \left[\frac{1}{2} + \frac{2025}{1352}\left(\int_0^1 (u')^2\, dx\right)^2\right]$
  - $\left[\int_0^x G(x,t)\, u dt + \int_x^1 G(x,t)\, u dt\right].$

The Picard's iteration gives u_0 the following algorithm.
Algorithm:
Choose initial $u_0(x)$, and repeat

$$\begin{aligned}
u_k(x) &= \int_0^x \left[G(x,t)\, A_k\right] dt + \int_x^1 \left[G(x,t)\, A_k\right] dt \\
&\quad -2\left(u_k'(1)\right) \left[\int_0^x G(x,t)\, t dt + \int_x^1 G(x,t)\, t dt\right] \\
&\quad + \left[\frac{1}{2} + \frac{2025}{1352}\left(\int_0^1 (u_k')^2\, dx\right)^2\right] \\
&\quad \left[\int_0^x G(x,t)\, u_{k-1} dt + \int_x^1 G(x,t)\, u_{k-1} dt\right] \left(\bmod\; x^{k+1}\right), \\
\text{where } A_k &= \left[\int_0^t G(t,s)\, f(s,u_{k-1}(s))\, ds + \int_t^1 G(t,s)\, f(s,u_{k-1}(s))\, ds\right].
\end{aligned}$$

Using $u_0(x) = 0$ as initial guess, then the numerical results are given below.

k	Error
10	0.0293802
20	0.0092330
30	0.0029322
40	0.0009361
50	0.0002994
60	0.0000958

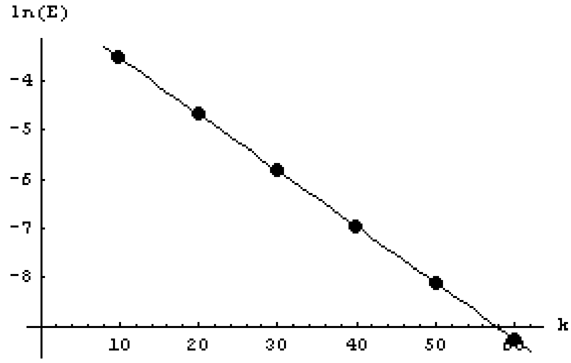

Figure 3.2

The asymptotic error formula is

$\ln E = -2.39192 - 0.114444k$

$\Rightarrow E = \left(e^{-2.39192}\right)\left(e^{-0.114444}\right)^k = 0.0914539 \times 0.891862^k.$

Finally, we consider the third boundary value problem as in [11]

$$u^{(4)}(x) - M\left(\int_0^L |u'(x)|^2 dx\right) u''(x) = f(x, u(x)),\ 0 < x < L, \tag{3.7}$$

$$u''(0) = u''(L) = 0, \tag{3.8}$$

$$u^{(3)}(0) - m\left(\int_0^L |u'(s)|^2 ds\right) u'(0) = g(u(0)), \tag{3.9}$$

$$u^{(3)}(L) - m\left(\int_0^L |u'(s)|^2 ds\right) u'(L) = h(u(L)), \tag{3.10}$$

where $M \in C(\mathbf{R}^+)$, $f \in C([0,L] \times \mathbf{R})$ and $g,\ h \in C(\mathbf{R})$ are real functions.

Given $f \in C[0,L]$, the quasilinear problem $(3.13-3.16)$ has a unique solution if and only if

$$\alpha = \frac{1}{L} M\left(\int_0^L |u'(s)|^2 ds\right)(u(0) - u(L)) + \frac{1}{L}\int_0^L (t-L) f(t)\, dt,$$

$$\beta = \frac{1}{L} M\left(\int_0^L |u'(s)|^2 ds\right)(u(0) - u(L)) + \frac{1}{L}\int_0^L t f(t)\, dt.$$

Suppose that $g,\ h$ are invertible. Then we can reduce our problem as before to the following system of two second order problems:

$$\begin{cases} u'' = M\left(||u'||_2^2\right) u + v, \\ u(0) = g^{-1}(\alpha) \\ u(L) = h^{-1}(\beta) \end{cases}$$

and

$$\begin{cases} v'' = f(x,u), \\ v(0) = -M\left(||u'||_2^2\right) u(0), \\ v(L) = -M\left(||u'||_2^2\right) u(L) \end{cases}$$

Therefore, using the Green's function, we see that a solution is given by

$$u(x) = \int_0^L G(x,t)\left[M\left(||u'||_2^2\right) u(t) + v(t)\right] dt + \frac{L-x}{L} g^{-1}(\alpha) + \frac{x}{L} h^{-1}(\beta),$$

where

$$v(t) = \int_0^L G(t,s) f(s, u(s))\, ds - \frac{L-t}{L} M\left(||u'||_2^2\right) u(0) - \frac{t}{L} M\left(||u'||_2^2\right) u(L).$$

The model problem is choose to be

$$f(x,u) = -30x^4 + 390x^2 - 60.$$

The bearings are represented by

$$g(s) = -4s,\ M(s) = \frac{1}{2} + \frac{231}{3800}s,\ h(s) = \frac{120}{7}s.$$

The exact solution in $[0,1]$ is

$$u(x) = x^6 - \frac{5}{2}x^4 + 4x + 1.$$

Some computational detail are as follows.

- $g^{-1}(\alpha) = \frac{-1}{4}\left[u'''(0) - M\left(\int_0^1 |u'(x)|^2 dx\right) u'(0)\right]$
- $h^{-1}(\beta) = \frac{7}{120}\left[u'''(1) - M\left(\int_0^1 |u'(x)|^2 dx\right) u'(1)\right]$
- $\int_0^1 G(x,t)\left[M\left(||u'||_2^2\right) u(t) + \int_0^1 G(t,s) f(s,u(s))\, ds\right] dt$
  $- \int_0^1 G(x,t)\left[\frac{L-t}{L} M\left(||u'||_2^2\right) u(0) - \frac{t}{L} M\left(||u'||_2^2\right) u(L)\right] dt$

$= M\left(||u'||_2^2\right) \int_0^1 G(x,t)\, u(t)\, dt + \int_0^1 G(x,t)\left[\int_0^1 G(t,s) f(s,u(s))\, ds\right] dt$
$-M\left(||u'||_2^2\right) u(0) \int_0^1 G(x,t)(1-t)\, dt - M\left(||u'||_2^2\right) u(1) \int_0^1 G(x,t)(t)\, dt$
Picard's method proposes algorithm:
Choose initial $u_0(x)$, and repeat

$$\begin{aligned} u_k(x) &= \frac{L-x}{L} \cdot \frac{-1}{4}\left[u_k'''(0) - A \cdot u_k'(0)\right] + \frac{x}{L} \cdot \frac{7}{120}\left[u_k'''(1) - A \cdot u_k'(1)\right] \\ &\quad + A \int_0^1 G(x,t)\, u_{k-1}(t)\, dt + \int_0^1 G(x,t)\left[\int_0^1 G(t,s) f(s, u_{k-1}(s))\, ds\right] dt \\ &\quad - A u_k(0) \int_0^1 G(x,t)(1-t)\, dt - A u_k(1) \int_0^1 G(x,t)\, t dt \ \left(\text{mod } x^{k+1}\right), \\ \text{where } A &= M\left(\int_0^1 |u_k'(x)|^2 dx\right). \end{aligned}$$

Using $u_0(x) = 0$ as initial guess, then the numerical results are given below.

k	Error
10	0.37003
20	0.02161
30	0.00133
40	8.2×10^{-5}
50	5.1×10^{-6}
60	3.1×10^{-7}

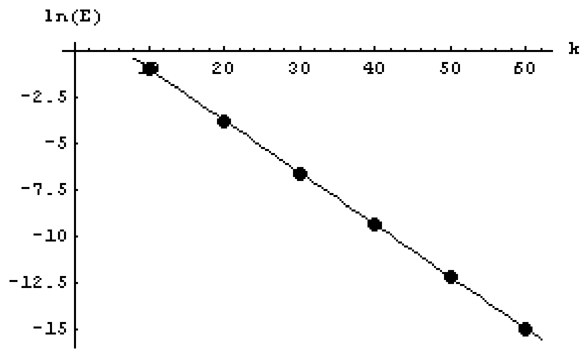

Figure 3.3

The asymptotic error formula is
$\ln E = 1.77487 - 0.27944k$
$\Rightarrow E = \left(e^{1.77487}\right)\left(e^{-0.27944}\right)^k = 5.89951 \times 0.756207^k.$

3.4 Finite Difference Method

In this section, we use finite difference method to solve the integral equation. Consider the first problem (3.6), it satisfies the integral equation

$$u(x) = \frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + Ax^3$$
$$+\{5A + \left[\frac{12}{27}\int_0^1 tf(t,u(t))dt\right]^{\frac{1}{3}} - \frac{1}{6}\int_0^1 (1-t)^3 f(t,u(t))dt\}x - 6A,$$

where $A = \frac{1}{6}\left[\int_0^1 (t-1)f(t,u(t))dt\right]$.

We consider $0 = x_0 < x_1 < \cdots < x_n = L$ be a discretization of the interval $[0, L]$ with mesh size $h = x_i - x_{i-1}$. Let $u_i = u(x_i)$, $f_i = f(x_i, u_i)$, then

- $\int_0^x (x-t)^3 f(t,u(t))dt \approx \sum_{k=1}^{i} h(x_i - x_k)^3 f(x_k, u_k)$.
- $\int_0^1 (t-1)f(t,u(t))dt \approx \sum_{k=1}^{n} h(x_k - 1) f(x_k, u_k)$.
- $\int_0^1 tf(t,u(t))dt \approx \sum_{k=1}^{n} hx_k f(x_k, u_k)$.
- $\int_0^1 (1-t)^3 f(t,u(t))dt \approx \sum_{k=1}^{n} h(1 - x_k)^3 f(x_k, u_k)$.

So the equation becomes:

$$u_i = \frac{1}{6}\sum_{k=1}^{i} h(x_i - x_k)^3 f(x_k, u_k) + A(x_i)^3 + \{5A + \left[\frac{12}{27}\sum_{k=1}^{n} hx_k f(x_k, u_k)\right]^{\frac{1}{3}}$$
$$-\frac{1}{6}\sum_{k=1}^{n} h(1 - x_k)^3 f(x_k, u_k)\}x_i - 6A,\ i = 1, \ldots, n,$$

where $A = \frac{1}{6}\left[\sum_{k=1}^{n} h(x_k - 1) f(x_k, u_k)\right]$.

Then we use Newton's method to solve this nonlinear system. The numerical results are given below.

n	Iteration numbers	Error
10	4	0.060798
20	4	0.031138
30	4	0.020922
40	4	0.015753
50	4	0.012632
60	4	0.010543

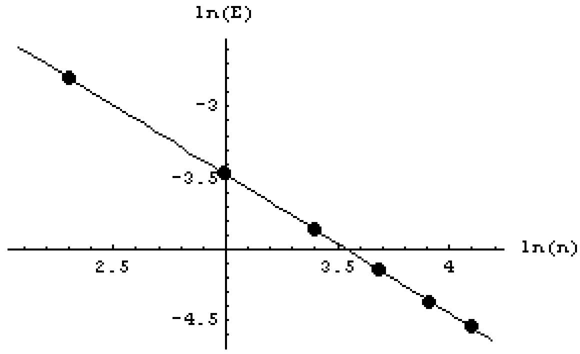

Figure 3.4

The asymptotic error formula is

$\ln E = -0.543507 - 0.978186 \ln n$

$\Rightarrow E = \frac{\exp^{-0.543507}}{n^{0.978186}} = \frac{0.580708}{n^{0.978186}} = 0.580708 \times h^{0.978186} \approx O\left(h^{1.0}\right) \rightarrow 0$, as $h \rightarrow 0$.

Next, we deal with the second problem, which equivalent integral equation is

$u(x) = \int_0^x \left[G(x,t) F(t)\right] dt + \int_x^1 \left[G(x,t) F(t)\right] dt$
$-2\left(u'(1)\right) \left[\int_0^x G(x,t)\, tdt + \int_x^1 G(x,t)\, tdt\right]$
$+ \left[\frac{1}{2} + \frac{2025}{1352} \left(\int_0^1 (u')^2\, dx\right)^2\right] \left[\int_0^x G(x,t)\, udt + \int_x^1 G(x,t)\, udt\right],$

where $F(t) = \int_0^t G(t,s) f(s, u(s))\, ds + \int_t^1 G(t,s) f(s, u(s))\, ds$.

Consider uniform discretization of $[0, L]$, $u_i = u(x_i)$ and $f_i = f(x_i, u_i)$ as before. Then

- $F(x_k) = \int_0^{x_k} G(x_k, s) f(s, u(s))\, ds + \int_{x_k}^1 G(x_k, s) f(s, u(s))\, ds$

  $= \int_0^{x_k} s(x_k - 1) f(s, u(s))\, ds + \int_{x_k}^1 x_k (s-1) f(s, u(s))\, ds$
  $= \sum_{t=0}^{k-1} hx_t (x_k - 1) f(x_t, u(x_t)) + \sum_{t=k}^{n-1} hx_k (x_t - 1) f(x_t, u(x_t))$.

- $\int_0^x G(x,t) F(t)\, dt = \int_0^x t(x-1) F(t)\, dt = \sum_{k=0}^{i-1} hx_k (x_i - 1) F(x_k)$.

- $u'(1) = \frac{u_n - u_{n-1}}{h}$.

- $\int_0^x G(x,t)\, tdt + \int_x^1 G(x,t)\, tdt$

$= \int_0^x t(x-1)\,tdt + \int_x^1 x(t-1)\,tdt$
$= (x_i - 1)\frac{(x_i)^3}{3} + \left(x_i\left(\frac{-1}{6}\cdot(x_i-1)^2(2x_i+1)\right)\right).$

- $\int_0^1 (u')^2\,dx = \sum\limits_{i=0}^{n-1} h\left[u'(x_i)\right]^2 = \sum\limits_{i=0}^{n-1} h\left(\frac{u_{i+1}-u_i}{h}\right)^2.$

- $\int_0^x G(x,t)\,udt + \int_x^1 G(x,t)\,udt$

$= \sum\limits_{k=0}^{i-1} hx_k(x_i-1)\,u(x_k) + \sum\limits_{k=i}^{n-1} hx_i(x_k-1)\,u(x_k).$

So the equation becomes:

$$u_i = \sum_{k=0}^{i-1} hx_k(x_i-1)F(x_k) + \sum_{k=i}^{n-1} hx_i(x_k-1)F(x_k)$$
$$-2\left(\frac{u_n-u_{n-1}}{h}\right)\left[(x_i-1)\frac{(x_i)^3}{3} + x_i\left(\frac{-1}{6}(x_i-1)^2(2x_i+1)\right)\right]$$
$$+\left[\frac{1}{2}+\frac{2025}{1352}\left(\sum_{i=0}^{n-1} h\left[\frac{u_{i+1}-u_i}{h}\right]^2\right)^2\right]\left[\sum_{k=0}^{i-1} hx_k(x_i-1)\,u(x_k) + \sum_{k=i}^{n-1} hx_i(x_k-1)\,u(x_k)\right],$$

$i = 1, \ldots, n$, where

$F(x_k) = \sum\limits_{t=0}^{k-1} hx_t(x_k-1)\,f(x_t, u(x_t)) + \sum\limits_{t=k}^{n-1} hx_k(x_t-1)\,f(x_t, u(x_t)).$

Then we use Newton's method to solve this nonlinear system. The numerical results are given below.

n	Iteration numbers	Error
10	3	0.005431
20	3	0.001753
30	3	0.000978
40	3	0.000663
50	3	0.000499
60	3	0.000397

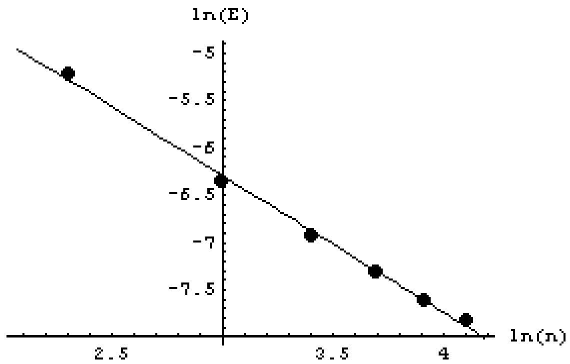

Figure 3.5

The asymptotic error formula is

$\ln E = -1.92009 - 1.45747 \ln n$

$\Rightarrow E = \frac{\exp^{-1.92009}}{n^{1.45747}} = 0.146594 \times h^{1.45747} \approx O\left(h^{1.5}\right) \to 0$, as $h \to 0$.

Finally, we deal with the third problem. Its integral equation is

$$u(x) = \int_0^1 G(x,t)\left[M\left(||u'||_2^2\right)u(t) + v(t)\right]dt + \frac{L-x}{L}g^{-1}(\alpha) + \frac{x}{L}h^{-1}(\beta),$$

where $v(t) = \int_0^1 G(t,s) f(s,u(s))\,ds - \frac{L-t}{L}M\left(||u'||_2^2\right)u(0) - \frac{t}{L}M\left(||u'||_2^2\right)u(L)$.

Discetization as above, we have

- $A = M\left(\int_0^1 |u'(x)|^2 dx\right) = \frac{1}{2} + \frac{231}{3800}\left(\sum\limits_{i=1}^{n} h\left(\frac{3u_i - 4u_{i-1} + u_{i-2}}{2h}\right)^2\right).$

- $\frac{L-x}{L}\frac{-1}{4}\left[u'''(0) - Au'(0)\right] = (1 - x_i) - \frac{1}{4}\left[\frac{-3u_4 + 14u_3 - 24u_2 + 18u_1 - 5u_0}{2h^3} - \frac{2u_3 - 9u_2 + 18u_1 - 11u_0}{6h}A\right].$

- $\frac{x}{L}\frac{7}{120}\left[u'''(1) - Au'(1)\right] = x_i\frac{7}{120}\left[\begin{array}{c}\frac{5u_n - 18u_{n-1} + 24u_{n-2} - 14u_{n-3} + 3u_{n-4}}{2h^3} \\ -\frac{11u_n - 18u_{n-1} + 9u_{n-2} - 2u_{n-3}}{6h}A\end{array}\right].$

- $A\int_0^1 G(x,t)u(t)\,dt = A\left[\left(\sum\limits_{k=1}^{i} h\left(x_k\left(x_i - 1\right)u\left(x_k\right)\right)\right) + \left(\sum\limits_{k=i+1}^{n} h\left(x_i\left(x_k - 1\right)u\left(x_k\right)\right)\right)\right],$

- $F(x_k) = \int_0^{x_k} G(x_k,s) f(s,u(s))\,ds + \int_{x_k}^1 G(x_k,s) f(s,u(s))\,ds$

$= \int_0^{x_k} s(x_k - 1) f(s,u(s))\,ds + \int_{x_k}^1 x_k(s-1) f(s,u(s))\,ds$

$= \sum\limits_{t=0}^{k-1} hx_t(x_k - 1) f(x_t, u(x_t)) + \sum\limits_{t=k}^{n-1} hx_k(x_t - 1) f(x_t, u(x_t)).$

- $\int_0^x G(x,t)\left[\int_0^1 G(t,s) f(s,u(s))\,ds\right]dt = \int_0^x G(x,t)F(t)\,dt$

$= \int_0^x t(x-1)F(t)\,dt = \sum\limits_{k=0}^{i-1} hx_k(x_i - 1)F(x_k).$

- $Au(0)\int_0^1 G(x,t)(1-t)\,dt = Au_0\left[\int_0^x t(x-1)(t-1)\,dt + \int_x^1 x(t-1)(t-1)\,dt\right],$

- $Au(1)\int_0^1 G(x,t)(t)\,dt = Au_n\left[\int_0^x t(x-1)(-t)\,dt + \int_x^1 x(t-1)(-t)\,dt\right].$

So we have the equation becomes:

$$u_i = \sum_{k=0}^{i-1} hx_k (x_i - 1) F(x_k) + \sum_{k=i}^{n-1} hx_i (x_k - 1) F(x_k) + (1 - x_i)$$
$$-\frac{1}{4}\left[\frac{-3u_4+14u_3-24u_2+18u_1-5u_0}{2h^3} - \frac{2u_3-9u_2+18u_1-11u_0}{6h}A\right]$$
$$+\frac{7}{120}x_i\left[\frac{5u_n-18u_{n-1}+24u_{n-2}-14u_{n-3}+3u_{n-4}}{2h^3} - \frac{11u_n-18u_{n-1}+9u_{n-2}-2u_{n-3}}{6h}A\right]$$
$$+A\left[\sum_{k=1}^{i} h(x_k(x_i-1)u(x_k)) + \sum_{k=i+1}^{n} h(x_i(x_k-1)u(x_k))\right]$$
$$-Au_0\left[\int_0^x t(x-1)(t-1)\,dt + \int_x^1 x(t-1)(t-1)\,dt\right]$$
$$-Au_n\left[\int_0^x t(x-1)(-t)\,dt + \int_x^1 x(t-1)(-t)\,dt\right],\ i = 1, \ldots, n,$$

where $A = M\left(\int_0^1 |u'(x)|^2 dx\right) = \frac{1}{2} + \frac{231}{3800}\sum_{i=1}^{n} h\left(\frac{3u_i-4u_{i-1}+u_{i-2}}{2h}\right)^2.$

Then we use Newton's method to solve this nonlinear system. The numerical results are given below.

n	Iteration numbers	Error
10	3	0.452612
20	4	0.147864
30	4	0.069239
40	4	0.039457
50	4	0.025221
60	4	0.017371

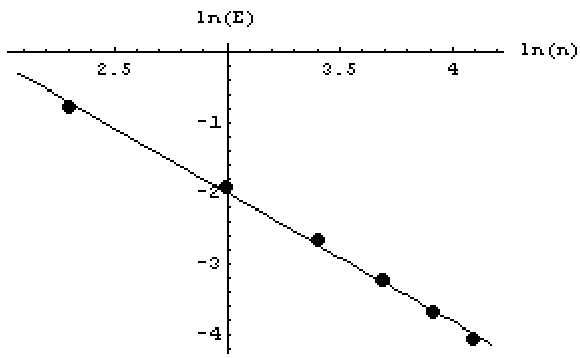

Figure 3.6

The asymptotic error formula is

$$\ln E = 3.4805 - 1.82512 \ln n$$
$$\Rightarrow E = \frac{\exp^{3.4805}}{n^{1.82512}} = \frac{12.1689}{n^{1.82512}} = 32.476 \times h^{1.82512} \approx O\left(h^{1.8}\right) \to 0, \text{ as } h \to 0.$$

3.5 Collocation Method

In this section, we use the collocation method to solve the integral equation. Consider the first problem $Lu = F$, where

$Lu = u(x) - \{\frac{1}{6}\int_0^x (x-t)^3 f(t,u(t))dt + Ax^3 + \{5A + \left[\frac{12}{27}\int_0^1 tf(t,u(t))dt\right]^{\frac{1}{3}} - \frac{1}{6}\int_0^1 (1-t)^3 f(t,u(t))dt\}x - 6A\} = 0$, where $A = \frac{1}{6}\left[\int_0^1 (t-1)f(t,u(t))dt\right]$.

Collocation method first chooses trial functions$\{\phi_k(x)\}$, and uses the numerical solution of the form

$$\widehat{u}(x) = \sum_{k=0}^{n} c_k \phi_k(x).$$

Then find the coefficients $\{c_k\}$ such that

$$L\widehat{u}(x_i) = F(x_i) \text{ for } \forall i = 1, \cdots, n.$$

To be more precise

$$\sum_{k=0}^{n} c_k \phi_k(x_i) = \frac{1}{6}\int_0^{x_i} (x_i - t)^3 f(t, \widehat{u}(t))dt + Ax_i^3$$
$$+\{5A + \left[\frac{12}{27}\int_0^1 tf(t,\widehat{u}(t))dt\right]^{\frac{1}{3}} - \frac{1}{6}\int_0^1 (1-t)^3 f(t,\widehat{u}(t))dt\}x_i - 6A,$$

$\forall i = 1, \cdots, n$, where $A = \frac{1}{6}\left[\int_0^1 (t-1)f(t, \sum_{k=0}^{n} c_k\phi_k(t))dt\right]$. Then we use Newton method to solve this nonlinear system for c_k.

For numerical experiment, let trial functions:

$$\{\phi_k(x)\} = \{\sin\left(\frac{\pi}{2}kx\right), \cos\left(\frac{\pi}{2}kx\right)\}_{k=0}^{n},$$

so

$$\widehat{u}(x) = \sum_{k=1}^{n} c_k \sin\left(\frac{\pi}{2}kx\right) + \sum_{k=0}^{n} c_{n+1+k}\cos\left(\frac{\pi}{2}kx\right).$$

The numerical results are given below, , where $n' = 2 \times n + 1$. Let $\varepsilon = 10^{-5}$.

n'	Iteration numbers	Error
3	4	0.0010266000
5	4	0.0000737476
7	4	4.9794×10^{-6}
9	4	3.3875×10^{-7}
11	4	2.3472×10^{-8}
13	4	1.6558×10^{-9}

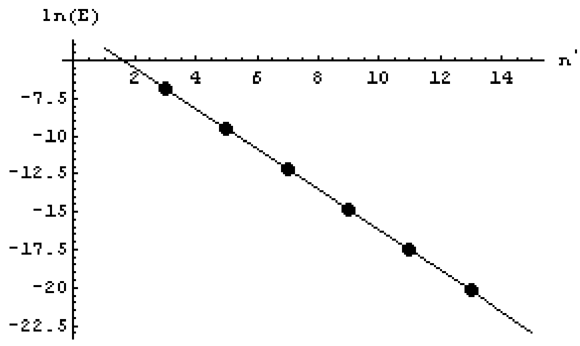

Figure 3.7

The asymptotic error formula is

$\ln E = -2.85902 - 1.33619^{n'}$

$\Rightarrow E = \left(e^{-2.85902}\right)\left(e^{-1.33619}\right)^{n'} = 0.0573249 \times 0.262845^{n'}$.

3.6 Weighted Residual Method

In this section, we use the weighted residual method to solve the integral equation. Consider the first problem $Lu = 0$, where $Lu = u(x) - \{\frac{1}{6}\int_0^x (x-t)^3 f(t, u(t))dt + Ax^3 + \{5A + \left[\frac{12}{27}\int_0^1 tf(t, u(t))dt\right]^{\frac{1}{3}} - \frac{1}{6}\int_0^1 (1-t)^3 f(t, u(t))dt\}x - 6A\} = 0$, and $A = \frac{1}{6}\left[\int_0^1 (t-1)f(t, u(t))dt\right]$.

First, choose $\{\phi_k(x)\}$ and $\{\Phi_i(x)\}$ as trial and test functions respectively. Consider the numerical solution to be

$$\widehat{u}(x) = \sum_{k=0}^{n} c_k \phi_k(x). \tag{3.11}$$

Weighted residual method find the coefficients c_k by the Galerkin condition

$$\int_0^L L(\widehat{u})\,\Phi_i dx = 0, \forall i = 1, \ldots, n. \tag{3.12}$$

Note that the Galerkin method uses $\{\phi_k(x)\} = \{\Phi_i(x)\}$.

Substituting (3.11) into (3.12), we have

$$\int_0^L \left(\begin{array}{c} \sum\limits_{k=0}^{n} c_k\phi_k(x) - \{\frac{1}{6}\int_0^x (x-t)^3 f(t, \sum\limits_{k=0}^{n} c_k\phi_k(t))dt + Ax^3 + \{5A \\ + \left[\frac{12}{27}\int_0^1 tf(t, \sum\limits_{k=0}^{n} c_k\phi_k(t))dt\right]^{\frac{1}{3}} - \frac{1}{6}\int_0^1 (1-t)^3 f(t, \sum\limits_{k=0}^{n} c_k\phi_k(t))dt\}x - 6A\} \end{array} \right) \Phi_i dx = 0,$$

$\forall i = 1, \cdots, n$, where $A = \frac{1}{6}\left[\int_0^1 (t-1)f(t, \sum\limits_{k=0}^{n} c_k\phi_k(t))dt\right]$. Similarly, Newton method is used to solve the nonlinear system for c_k.

For numerical experiment, choose trial function

$$\{\phi_k(x)\} = \{\sin\left(\tfrac{\pi}{2}kx\right), \cos\left(\tfrac{\pi}{2}kx\right)\}_{k=0}^{n},$$

and test function

$$\{\Phi_i\} = \{\sin(i\pi x)\}_{i=1}^{2n+1}.$$

So the numerical solution

$$\widehat{u}(x) = \sum_{k=1}^{n} c_k \sin\left(\tfrac{\pi}{2}kx\right) + \sum_{k=0}^{n} c_{n+1+k}\cos\left(\tfrac{\pi}{2}kx\right).$$

The numerical results are given below, where $n' = 2 \times n+1$. Let $\varepsilon = 10^{-5}$.

n'	Iteration numbers	Error
3	4	0.00357158
5	4	0.00030526
7	4	0.00002235
9	4	1.5285×10^{-6}
11	4	1.0117×10^{-7}

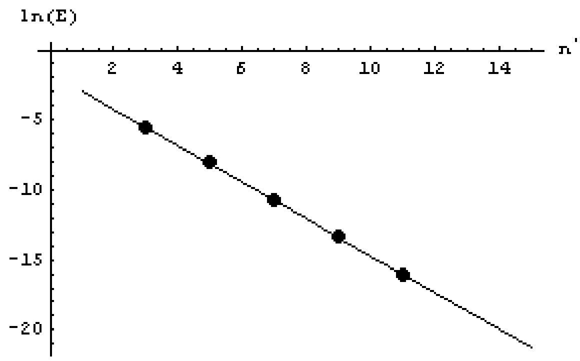

Figure 3.8

The asymptotic error formula is

$\ln E = -1.60298 - 1.31202^{n'}$

$\Rightarrow E = \left(e^{-1.60298}\right)\left(e^{-1.31202}\right)^{n'} = 0.201296 \times 0.269276^{n'}$.

Compare their rate of convergence, we conclude that the combination of Newton's and Weighted Residual Methods is the fastest among all the method in this chapter.

3.7 Conclusion

From our previous numerical experiments, we know Weighted Residual Method including Spectral and Collocation methods is the best numerical methods to deal with these nonlinear beam problems. Although some of them may diverge if the test and trial functions are not appropriate by chosen. But a good basis function will make the approximation converge exponentially.

As mentioned previously, the beam problem is extremely practical, and it has great contribution to the engineering if we know its best numerical algorithm. The theory of the well-posedness of these boundary value problems can help us to understand deeply their nature. Our numerical method deals with more complex boundary value problems and can be applied to the other nonlinear problems. Our technique can also be extended to solve the beam vibration problem, which is a partial differential equation, e.g. [9].

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